A UNIFIED THEORY FOR CONTINUOUS IN TIME EVOLVING FINITE ELEMENT SPACE APPROXIMATIONS TO PARTIAL DIFFERENTIAL EQUATIONS IN EVOLVING DOMAINS

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ABSTRACT. We develop a unified theory for continuous in time finite element discretisations of partial differential equations posed in evolving domains including the consideration of equations posed on evolving surfaces and bulk domains as well coupled surface bulk systems. We use an abstract variational setting with time dependent function spaces and abstract time dependent finite element spaces. Optimal a priori bounds are shown under usual assumptions on perturbations of bilinear forms and approximation properties of the abstract finite element spaces. The abstract theory is applied to evolving finite elements in both flat and curved spaces allowing the approximation of parabolic equations in general domains. Numerical experiments are described which confirm the rates of convergence.

1. INTRODUCTION

In this paper, we develop a unified theory for finite element discretisations of partial differential equations posed in evolving domains including the consideration of equations posed on evolving surfaces and bulk domains as well coupled surface bulk systems. The discretisation is based on evolving finite element spaces defined on evolving triangulations using isoparametric elements. Optimal order a priori error bounds are proven. This unification is achieved by using an abstract variational setting with time dependent abstract function spaces and time dependent abstract finite element spaces. Given a Hilbert space triple

$$\mathcal{V}(t) \subset \mathcal{H} \subset \mathcal{V}^*(t),$$

the abstract strong formulation is: Find $u(t) \in \mathcal{V}(t)$ such that

$$\partial^\bullet u + L(t)u + \omega(t)u = 0 \qquad \text{in } \mathcal{V}^*(t) \tag{1.1a}$$

$$u(0) = u_0. \tag{1.1b}$$

where $\mathcal{V}(t)$ is an appropriate time dependent Hilbert space with dual $\mathcal{V}^*(t)$, $\partial^\bullet u$ is an appropriate abstract material derivative arising from the parameterisation, $L(t)$ is an elliptic operator satisfying suitable coercivity properties and $\omega(t)$ is a lower term arising from evolution of the domain. Similar to the case of time independent function spaces this equation may be written in variational form as

Problem 1.1. *Given $u_0 \in \mathcal{V}_0$, find $u \in L^2_{\mathcal{V}}$ with $\partial^\bullet u \in L^2_{\mathcal{H}}$ such that for almost every $t \in [0,T]$,*

$$m(t;\partial^\bullet u,\varphi) + g(t;u,\varphi) + a(t;u,\varphi) = 0 \qquad \textit{for all } \varphi \in \mathcal{V}(t), \tag{1.2}$$

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subject to the initial condition $u(\cdot,0) = u_0$.

Here $L^2_{\mathcal{V}}$ and $L^2_{\mathcal{H}}$ are generalisations of the Bochner spaces $L^2(0,T;V)$ and $L^2(0,T;H)$ in the case of time independent spaces. The bilinear form $a(\cdot,\cdot)$ is associated with the elliptic operator L and the bilinear forms m and g are associated with the $\mathcal{H}(t)$-inner product and its time derivative.

We formulate and analyse an abstract finite element discretisation based on a Galerkin ansatz with perturbations of the bilinear forms. Under assumptions on the approximation of geometry and the approximation of function spaces by abstract finite element spaces optimal order error bounds are proved. Evolving finite element spaces for bulk and surfaces are constructed. These are based on evolving Lagrange isoparametric finite elements.

This approach is applied to three model problems: a linear parabolic problem on an evolving compact n-dimensional surface embedded in $\mathbb{R}^{n+1}$, a linear parabolic problem in an evolving, bounded bulk domain in $\mathbb{R}^{n+1}$ and a linear parabolic problem coupling problems in an evolving, bounded bulk domain in $\mathbb{R}^{n+1}$ to a problem on its boundary. In each case, we assume that the evolution of the problem domain is prescribed. The abstract approach is applicable to other situations including dynamic boundary conditions (Kovacs and Lubich, 2016).

1.1. **Some partial differential equations.** Let $T > 0$. For $t \in [0,T]$ let $\Omega(t)$ denote an $(n+1)$-dimensional bounded, open, connected domain in $\mathbb{R}^{n+1}$, for $n = 2,3$. We denote by $\Gamma(t)$ the boundary of $\Omega(t)$ and assume that $\Gamma(t)$ is a compact, smooth hypersurface. We write $\Omega_0 = \Omega(0)$, $\Gamma_0 = \Gamma(t)$ and $\nu(\cdot,t)$ the normal to $\Gamma(t)$. We assume that $\Omega(t)$ is given by a parametrisation $G\colon \bar{\Omega}_0 \times [0,T] \to \mathbb{R}^{n+1}$ such that $\Omega(t) = G(\Omega_0,t)$ and $\Gamma(t) = G(\Gamma_0,t)$. We write w for the velocity defined by $w(G(\cdot),t) = \frac{d}{dt}G(\cdot,t)$. We will write G in terms of a flow $\Phi_t\colon \bar{\Omega}_0 \to \bar{\Omega}(t)$, with inverse $\Phi_{-t}\colon \bar{\Omega}(t) \to \bar{\Omega}_0$, given by $\Phi_t(\cdot) = G(\cdot,t)$. We write $\partial^\bullet$ for a material derivative and ∇_Γ and Δ_Γ for surface gradient and the Laplace-Beltrami operator. Section 4.1 gives precise definitions.

Remark 1.2. Note that velocity field w is the velocity of the parametrisation. In order to define the evolution of the domain we need only specify the normal velocity for a compact hypersurface and for a sub-manifold of a compact hypersurface we also specify the conormal velocity of the boundary. In particular we may use such a velocity in order to achieve a mesh with good properties as in the Arbitrary Lagrangian–Eulerian approach. Observe also that in the model from which the equation arises there may be a physical advective velocity which transports material. Care should be taken to distinguish between these velocities where necessary. Section 4.3 gives more details of the choice of the velocity field w and an example of how we may derive these partial differential equations.

First, we seek a time-dependent scalar surface field u such that

$$\partial^\bullet u + u \nabla_\Gamma \cdot w - L_\Gamma u = 0 \qquad \text{on } \Gamma(t) \tag{1.3a}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Gamma_0 := \Gamma(0), \tag{1.3b}$$

where L_Γ is the operator given by

$$L_\Gamma u := \nabla_\Gamma \cdot (\mathscr{A} \nabla_\Gamma u) + \nabla_\Gamma \cdot (\mathscr{B} u) + \mathscr{C} u, \tag{1.4}$$

where $\mathscr{A}$ is a smooth diffusion tensor which maps the tangent space of Γ into itself, $\mathscr{B}$ is a smooth tangential vector field and $\mathscr{C}$ is a smooth scalar field.

Second, we consider a problem in an evolving Cartesian bulk domain to find a time-dependent scalar field u such that

$$\partial^\bullet u + u\nabla\cdot w - L_\Omega u = 0 \qquad \text{on } \Omega(t) \tag{1.5a}$$

$$(\mathscr{A}\nabla u + \mathscr{B}u)\cdot \nu = 0 \qquad \text{on } \Gamma(t) \tag{1.5b}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Omega_0 := \Omega(0), \tag{1.5c}$$

where L_Ω is the operator given by

$$L_\Omega u := \nabla\cdot(\mathscr{A}\nabla u) + \nabla\cdot(\mathscr{B}u) + \mathscr{C}u, \tag{1.6}$$

where $\mathscr{A}$ is a smooth diffusion tensor, $\mathscr{B}$ a smooth vector field and $\mathscr{C}$ a smooth scalar field. We use the same notation $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ as in the surface case for simplicity.

Finally, we seek a time-dependent pair (u,v) with u a scalar volumetric field and v a scalar surface field such that

$$\partial^\bullet u + u\nabla\cdot w - L_\Omega u = 0 \qquad \text{on } \Omega(t) \tag{1.7a}$$

$$(\mathscr{A}_\Omega\nabla u + \mathscr{B}_\Omega u)\cdot\nu + (\alpha u - \beta v) = 0 \qquad \text{on } \Gamma(t) \tag{1.7b}$$

$$\partial^\bullet v + v\nabla_\Gamma\cdot w - L_\Gamma v - (\mathscr{A}_\Omega\nabla u + \mathscr{B}_\Omega u)\cdot\nu = 0 \qquad \text{on } \Gamma(t) \tag{1.7c}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Omega_0 := \Omega(0) \tag{1.7d}$$

$$v(\cdot,0) = v_0 \qquad \text{on } \Gamma_0 := \Gamma(0), \tag{1.7e}$$

where L_Ω, L_Γ are operators given above by (1.4) and (1.6). We consider (1.7b) as defining the flux between the domain $\Omega(t)$ and its boundary $\Gamma(t)$.

Remark 1.3. All the theory presented in this paper will be applicable with the addition of right hand side functions for each of these equations under appropriate assumptions on the data.

1.2. **Background.** Partial differential equations posed on complex evolving domains arise in numerous settings such as surfactant transport on fluid interfaces, receptor ligand dynamics on cell surfaces and phase separation on dissolving alloy surfaces (Deckelnick, Elliott, Kornhuber, and Sethian, 2015; Elliott, Ranner, and Venkataraman, 2017; Barrett, Garcke, and Nurnberg, 2015). Numerical approaches to solve these problems include surface finite elements, implicit surface formulations, diffuse interface approximations, trace finite elements and unfitted finite elements. See the works of Dziuk (1988); Dziuk and Elliott (2007); Deckelnick, Dziuk, Elliott, and Heine (2009); Dziuk and Elliott (2010); Deckelnick, Elliott, and Ranner (2014); Olshanskii and Reusken (2016); Burman, Hansbo, Larson, and Zahedi (2016) and the review of Dziuk and Elliott (2013a).

1.3. **Outline.** In Section 2, we introduce the abstract functional analytic setting in which we pose the continuous partial differential equations. An abstract analysis of evolving finite element methods is provided in Section 3. In Section 4.1, we formulate our descriptions of evolving domains necessary to pose the parabolic PDEs and the evolving finite element spaces. Section 5 deals with the construction of evolving surface finite element spaces and evolving discrete hypersurfaces. Sections 6–8 apply these ideas to tackle three model problems. The paper concludes with Section 9 where results of numerical experiments are given which confirm the proven error bounds.

2. ABSTRACT FORMULATION

2.1. **Evolving function spaces.** We introduce an abstract functional analytic setting derived by Alphonse, Elliott, and Stinner (2015a) based on the surface PDE setting of Vierling (2014). Using this formulation, we can pose partial differential equations on evolving domains in a fully rigorous setting. One of the key novelties of this work is to provide the basic theory for evolving Bochner-like spaces for evolving Hilbert spaces such as $H^1(\Gamma(t))$ in order make a definition similar to "$L^2(0,T;H^1(\Gamma(t)))$".

The work of Alphonse et al. (2015a) uses a Lagrangian formulation where the evolving domain is parametrised over the initial domain. This matches well with the arbitrary Langrangian-Eulerian finite element methods we will consider. A different functional analytic setting maybe more appropriate for different discretisation approaches such as the trace finite element method (Olshanskii, Reusken, and Xu, 2014; Olshanskii and Reusken, 2016) or the implicit surface approach (Dziuk and Elliott, 2010).

Definition 2.1 (Compatibility). For $t \in [0,T]$, let $\mathcal{X}(t)$ be a separable Hilbert space and denote by $\mathcal{X}_0 := \mathcal{X}(0)$. Let $\phi_t \colon \mathcal{X}_0 \to \mathcal{X}(t)$ be a family of invertible linear homeomorphisms, with inverse $\phi_{-t} \colon \mathcal{X}(t) \to \mathcal{X}_0$, such that there exists $C_{\mathcal{X}} > 0$ such that

$$\begin{aligned} \|\phi_t \eta\|_{\mathcal{X}(t)} &\le C_{\mathcal{X}} \|\eta\|_{\mathcal{X}(t)} && \text{for all } \eta \in \mathcal{X}_0 \\ \|\phi_{-t} \eta\|_{\mathcal{X}(t)} &\le C_{\mathcal{X}}^{-1} \|\eta\|_{\mathcal{X}(t)} && \text{for all } \eta \in \mathcal{X}(t), \end{aligned}$$

and such that the map $t \mapsto \|\phi_t \eta\|_{\mathcal{X}(t)}$ is continuous for all $\eta \in \mathcal{X}_0$. Under these circumstances, we call the pair $(\mathcal{X}(t), \phi_t)_{t\in[0,T]}$ *compatible*. We call the map ϕ_t the *push-forward* operator and ϕ_{-t} the *pull-back* operator.

Definition 2.2 (Evolving Hilbert triple). For each $t \in [0,T]$, let $\mathcal{V}(t)$ and $\mathcal{H}(t)$ be real, separable Hilbert spaces with $\mathcal{V}_0 := \mathcal{V}(0)$ and $\mathcal{H}_0 := \mathcal{H}(0)$ such that inclusion $\mathcal{V}(t) \subset \mathcal{H}(t)$ is continuous and dense. We will write $\|\cdot\|_{\mathcal{V}(t)}$ and $\|\cdot\|_{\mathcal{H}(t)}$ for the norms on $\mathcal{V}(t)$ and $\mathcal{H}(t)$, $(\cdot,\cdot)_{\mathcal{H}(t)}$ for the inner product on $\mathcal{H}(t)$ and $\langle \cdot,\cdot \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}$ for the pairing of $\mathcal{V}(t)$ with its dual. Let there exist a family of linear homeomorphisms $\phi_t \colon \mathcal{H}_0 \to \mathcal{H}(t)$ such that $(\mathcal{H}(t), \phi_t)_{t\in[0,T]}$ and $(\mathcal{V}(t), \phi_t|_{\mathcal{V}_0})_{t\in[0,T]}$ are compatible. We will write ϕ_t for $\phi_t|_{\mathcal{V}_0}$ also. It follows that $\mathcal{H}(t) \subset \mathcal{V}^*(t)$ continuously and densely. Under these assumptions, we say that $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))_{t\in[0,T]}$ is an *evolving Hilbert triple*.

For a compatible pair, we can define an equivalent structure to Bochner spaces in an evolving context. For $(\mathcal{X}(t), \phi_t)_{t\in[0,T]}$ a compatible pair, we define $L^2_{\mathcal{X}}$ to be

$$L^2_{\mathcal{X}} := \left\{ \eta \colon [0,T] \to \bigcup_{t\in[0,T]} \mathcal{X}(t) \times \{t\}, t \mapsto (\bar\eta(t), t) : \phi_{-\cdot} \bar\eta(\cdot) \in L^2(0,T;\mathcal{X}_0) \right\},$$

with norm

$$\|\eta\|_{L^2_{\mathcal{X}}} := \left(\int_0^T \|\bar\eta\|_{\mathcal{X}(t)}^2 \,\mathrm{d}t \right)^{\frac{1}{2}}.$$

One can show that the space $L^2_{\mathcal{X}}$ is a separable Hilbert space (Alphonse et al., 2015a, Corollary 2.12), $L^2_{\mathcal{X}}$ is isomorphic to $L^2(0,T;\mathcal{X}_0)$ and

$$C_{\mathcal{X}}{}^{-1} \|\eta\|_{L^2_{\mathcal{X}}} \le \left\| \phi_{-(\cdot)} \eta \right\|_{L^2(0,T;\mathcal{X}_0)} \le C_{\mathcal{X}} \|\eta\|_{L^2_{\mathcal{X}}} \qquad \text{for all } \eta \in L^2_{\mathcal{X}}.$$

Remark 2.3. If $(\mathcal{X}(t), \phi_t^{(1)})_{t\in[0,T]}$ and $(\mathcal{X}(t), \phi_t^{(2)})_{t\in[0,T]}$ are both compatible pairs then the spaces $L^2_{\mathcal{X}}$ induced using each push forward map have distinct but equivalent norms.

For $k \geq 0$, we also define the space of smoothly evolving in time functions by

$$C^k_{\mathcal{X}} := \left\{ \eta \in L^2_{\mathcal{X}} : t \mapsto \phi_{-t}\eta(\cdot,t) \in C^k([0,T],\mathcal{X}_0) \right\}$$
$$\mathcal{D}_{\mathcal{X}}(0,T) := \left\{ \eta \in L^2_{\mathcal{X}} : t \mapsto \phi_{-t}\eta(\cdot,t) \in \mathcal{D}((0,T);\mathcal{X}_0) \right\},$$

where $\mathcal{D}((0,T),\mathcal{X}_0)$ is the space of $\mathcal{X}_0$-valued infinitely differentiable functions compactly supported in the interval $(0,T)$.

For $\eta \in C^1_{\mathcal{H}}$, we can define a strong material derivative which we denote by $\partial^\bullet \eta \in C^0_{\mathcal{H}}$ by

$$\partial^\bullet \eta := \phi_t \left(\frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t}\eta) \right). \tag{2.1}$$

This is a temporal derivative which takes into account that fact that $\mathcal{H}(t)$ is changing as well as the function η.

We can further extend this definition to a weak material derivative. We will impose that an integration by parts in time formula holds. This is often called a transport formula because it takes into account the evolution of the space $\mathcal{H}(t)$ also. In order to provide this definition, we require a further assumption on $\mathcal{H}(t)$.

Assumption 2.4. We shall assume the following for all $\eta_0, \varphi_0 \in \mathcal{H}_0$:

$$\theta(t,\eta_0) := \frac{\mathrm{d}}{\mathrm{d}t} \|\phi_t \eta_0\|^2_{\mathcal{H}(t)} \text{ exists classically}$$
$$\eta_0 \mapsto \theta(t,\eta_0) \text{ is continuous}$$
$$|\theta(t,\eta_0+\varphi_0) - \theta(t;\eta_0-\varphi_0)| \leq c\,\|\eta_0\|_{\mathcal{H}_0}\|\varphi_0\|_{\mathcal{H}_0},$$

with the constant c independent of $t \in [0,T]$.

We define $\hat{g}(t;\cdot,\cdot)\colon \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}$ by

$$\hat{g}(t;\eta_0,\varphi_0) := \frac{1}{4}\big(\theta(t,\eta_0+\varphi_0) - \theta(t,\eta_0-\varphi_0)\big).$$

Then we have a bilinear form $g(t;\cdot,\cdot)\colon \mathcal{H}(t)\times\mathcal{H}(t) \to \mathbb{R}$ by

$$g(t;\eta,\varphi) := \hat{g}(t;\phi_{-t}\eta,\phi_{-t}\varphi).$$

It can be shown that the map $t \mapsto g(t;\eta,\varphi)$ is measurable for $\eta,\varphi \in L^2_{\mathcal{H}}$ and we have the following bound independently of t:

$$|g(t;\eta,\varphi)| \leq c\,\|\eta\|_{\mathcal{H}(t)}\|\varphi\|_{\mathcal{H}(t)}. \tag{2.2}$$

We say a function $\eta \in L^2_{\mathcal{V}}$ has a weak material derivative $\partial^\bullet \eta \in L^2_{\mathcal{V}^*}$ if

$$\int_0^T \langle \partial^\bullet \eta, \varphi \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}\,\mathrm{d}t = \int_0^T -(\eta,\partial^\bullet\varphi)_{\mathcal{H}(t)} + g(t;\eta,\varphi)\,\mathrm{d}t$$

for all $\varphi \in \mathcal{D}_V(0,T)$.

Lemma 2.5 (Abstract transport formula)**.** *For all $\eta,\varphi \in L^2_{\mathcal{V}}$ with weak material derivatives $\partial^\bullet\eta, \partial^\bullet\varphi \in L^2_{\mathcal{V}^*}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta,\varphi)_{\mathcal{H}(t)} = \langle \partial^\bullet\eta,\varphi\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \langle \partial^\bullet\varphi,\eta\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + g(t;\eta,\varphi). \tag{2.3}$$

Proof. See Alphonse et al. (2015a, Theorem 2.40). □

2.2. **Abstract formulation of the partial differential equation.** Let $T > 0$. We assume we are in the setting that we have an evolving Hilbert triple $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))_{t\in[0,T]}$ and Assumption 2.4 holds so that we have material derivative, which we denote, e.g. $\partial^\bullet \eta$ for appropriate η, and a transport formula for the $\mathcal{H}(t)$-inner product.

We assume that we have three time dependent bilinear forms m, g, a

$$m(t;\cdot,\cdot)\colon \mathcal{H}(t)\times\mathcal{H}(t)\to\mathbb{R}$$
$$g(t;\cdot,\cdot)\colon \mathcal{H}(t)\times\mathcal{H}(t)\to\mathbb{R}$$
$$a(t;\cdot,\cdot)\colon \mathcal{V}(t)\times\mathcal{V}(t)\to\mathbb{R}.$$

We consider problems of the following form:

Problem 2.6. *Given $u_0 \in \mathcal{V}_0$, find $u \in L^2_{\mathcal{V}}$ with $\partial^\bullet u \in L^2_{\mathcal{H}}$ such that for almost every $t \in [0,T]$,*

$$m(t;\partial^\bullet u,\varphi) + g(t;u,\varphi) + a(t;u,\varphi) = 0 \qquad \text{for all } \varphi \in \mathcal{V}(t), \tag{2.4}$$

subject to the initial condition $u(\cdot,0) = u_0$.

Remark 2.7. The abstract formulation of Alphonse et al. (2015a) allows for a weaker formulation with initial condition in $\mathcal{H}_0$ but we do not wish to consider such solutions here.

In order to make sense of this formulation we restrict to the following assumptions on the bilinear forms.

Assumptions on m: First, we assume that $m(t;\cdot,\cdot)$ is symmetric:

$$m(t;\eta,\varphi) = m(t;\varphi,\eta) \qquad \text{for } \eta,\varphi \in H(t). \tag{M1}$$

We assume that there exists $c_1, c_2 > 0$ such that for all $t \in [0,T]$, we have

$$c_1 \|\eta\|_{\mathcal{H}(t)} \le (m(t;\eta,\eta))^{1/2} \le c_2 \|\eta\|_{\mathcal{H}(t)} \qquad \text{for all } \eta \in \mathcal{H}(t). \tag{M2}$$

Assumptions on g: We assume the existence of a bilinear form $g(t;\cdot,\cdot)$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m(t;\eta,\eta) = m(t;\partial^\bullet\eta,\eta) + \frac{1}{2} g(t;\eta,\eta) \qquad \text{for } \eta \in C^1_{\mathcal{H}}, \tag{G1}$$

such that there exists $c_3 > 0$ such that for all $t \in [0,T]$

$$|g(t;\eta,\varphi)| \le c_3 \|\eta\|_{\mathcal{H}(t)} \|\varphi\|_{\mathcal{H}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{H}(t). \tag{G2}$$

Assumptions on a: We assume that the map

$$t \mapsto a(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in L^2_{\mathcal{V}} \tag{A1}$$

is measurable, and there exists $c_4, c_5, c_6 > 0$ such that for all $t \in [0,T]$ we have

$$a(t;\eta,\eta) \ge c_4 \|\eta\|^2_{\mathcal{V}(t)} - c_5 \|\eta\|^2_{\mathcal{H}(t)} \qquad \text{for } \eta \in \mathcal{V}(t) \tag{A2}$$

$$|a(t;\eta,\varphi)| \le c_6 \|\eta\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{V}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{V}(t). \tag{A3}$$

Finally, we assume the existence of a bilinear form $b(t;\cdot,\cdot)\colon \mathcal{V}(t)\times\mathcal{V}(t)\to\mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t;\eta,\varphi) = a(t;\partial^\bullet\eta,\varphi) + a(t;\eta,\partial^\bullet\varphi) + b(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{V}(t)}, \tag{B1}$$

and that there exists $c_8 > 0$ such that for all $t \in [0,T]$ we have

$$|b(t;\eta,\varphi)| \le c_8 \|\eta\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{V}(t)}. \tag{B2}$$

Remark 2.8. We allow for the case that a is non-symmetric. This is consistent with the choice of Alphonse et al. (2015a) but in contrast to much of the finite element literature (e.g. Elliott and Venkataraman (2015) use different bilinear forms for diffusion and advection terms in a parabolic operator).

Theorem 2.9. *Let Assumptions (M1), (M2), (G1), (G2), (A1), (A2), (A3), (B1) and (B2) hold. The continuous problem (2.4) has a unique solution $u \in L^2_{\mathcal{V}}$ with $\partial^\bullet u \in L^2_{\mathcal{H}}$ which satisfies the stability bound*

$$\int_0^T \|u\|^2_{\mathcal{V}(t)} + \|\partial^\bullet u\|^2_{\mathcal{H}(t)} \,\mathrm{d}t \leq c \|u_0\|^2_{\mathcal{V}_0}. \tag{2.5}$$

Proof. The proofs follows by a Galerkin argument. The proof is very similar to Alphonse et al. (2015a, Theorem 3.6 and 3.13). We do not show the details here. ☐

Note that if we make the stronger assumption on g that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m(t;\eta,\eta) = m(t;\partial^\bullet \eta,\eta) + \frac{1}{2} g(t;\eta,\eta) \qquad \text{for } \eta \in L^2_{\mathcal{V}} \text{ with } \partial^\bullet \eta \in L^2_{\mathcal{H}}, \tag{G1'}$$

then u also satisfies the variational form of (2.4)

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t;u,\varphi) + a(t;u,\varphi) = m(t;\partial^\bullet \varphi) \qquad \text{for } \varphi \in L^2_{\mathcal{V}} \text{ with } \partial^\bullet \varphi \in L^2_{\mathcal{H}}. \tag{2.6}$$

3. Abstract discretisation analysis

3.1. **Abstract formulation of the discrete problem.** Let $T > 0$ and $h \in (0,h_0)$. Let $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ be a collection of finite dimensional spaces equipped with two norms $\|\cdot\|_{\mathcal{H}_h(t)}$ and $\|\cdot\|_{\mathcal{V}_h(t)}$ for which there exists a constant $c > 0$ such that for all $h \in (0,h_0)$ and all $t \in (0,T)$, we have

$$\|\chi_h\|_{\mathcal{H}_h(t)} \leq \|\chi_h\|_{\mathcal{V}_h(t)} \qquad \text{for all } \chi_h \in V_h(t). \tag{3.1}$$

We assume that there exists a discrete push forward map $\phi_t^h : \mathcal{V}_h(0) \to \mathcal{V}_h(t)$ such that $\{V_h(t), \phi_t^h\}_{t\in[0,T]}$ is, uniformly with respect to h, a compatible pair. That is there exists $C_1, C_2 > 0$ such that, for all $h \in (0,h_0)$,

$$\begin{aligned} C_1^{-1} \|\chi_h\|_{\mathcal{V}_h(0)} &\leq \left\|\phi_t^h \chi_h\right\|_{\mathcal{V}_h(t)} \leq C_1 \|\chi_h\|_{\mathcal{V}_h(0)} && \text{for all } \chi_h \in \mathcal{V}_h(0) \\ C_2^{-1} \|\chi_h\|_{\mathcal{H}_h(0)} &\leq \left\|\phi_t^h \chi_h\right\|_{\mathcal{H}_h(t)} \leq C_2 \|\chi_h\|_{\mathcal{H}_h(0)} && \text{for all } \chi_h \in \mathcal{V}_h(0). \end{aligned}$$

We will write $\partial_h^\bullet \chi_h$ for the material derivative with respect to the push-forward map ϕ_t^h. Let m_h and a_h be two time dependent bilinear forms:

$$\begin{aligned} m_h(t;\cdot,\cdot)&\colon \mathcal{V}_h(t) \times \mathcal{V}_h(t) \to \mathbb{R} \\ a_h(t;\cdot,\cdot)&\colon \mathcal{V}_h(t) \times \mathcal{V}_h(t) \to \mathbb{R}. \end{aligned}$$

Motivated by the variational form (2.6), we consider semi-discrete problems of the following form:

Problem 3.1. *Given $U_0 \in \mathcal{V}_{h,0}$, find $U_h \in C^1_{\mathcal{V}_h}$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;U_h,\phi_h) + a_h(t;U_h,\phi_h) = m_h(t;U_h,\partial_h^\bullet \phi_h) \qquad \textit{for all } \phi_h \in C^1_{\mathcal{V}_h}. \tag{3.2}$$

Let the dimension of $\mathcal{V}_h(t)$ be N for all $t \in [0,T]$. We write $\{\chi_i(\cdot,0)\}_{i=1}^N$ for a basis of $\mathcal{V}_h(0)$ and push-forward to construct a time dependent basis $\{\chi_i(\cdot,t)\}_{i=1}^N$ of $\mathcal{V}_h(t)$ by

$$\chi_i(\cdot,t) = \phi_t^h(\chi_i(\cdot,0)).$$

This implies that $\partial_h^\bullet \chi_i = 0$.

We will seek a solution U_h using a decomposition into the time dependent basis functions $\{\chi_i\}_{i=1}^N$ of $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ and write

$$U_h(x,t) = \sum_{i=1}^N \alpha_i(t)\chi_i(x,t) \qquad \text{for } x \in \Gamma_h(t), \tag{3.3}$$

where $\alpha(t) = (\alpha_1(t),\ldots,\alpha_N(t)) \in \mathbb{R}^N$. Using this notation (3.2) is equivalent to finding $\alpha \in C^1([0,T];\mathbb{R}^N)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{M}(t)\alpha(t)\big) + \mathcal{S}(t)\alpha(t) = 0, \tag{3.4}$$

where

$$\mathcal{M}(t)_{ij} = m_h(t;\chi_i,\chi_j) \qquad \mathcal{S}(t)_{ij} = a_h(t;\chi_i,\chi_j) \qquad \text{for } i,j = 1,\ldots,N.$$

We have used the fact here that $\partial_h^\bullet \chi_i = 0$ for $1 \le i \le N$.

Remark 3.2. Here, we are thinking of the case that m_h approximates the m bilinear form and a_h approximates the a bilinear form, $\mathcal{V}_h(t)$ approximates the space $\mathcal{V}(t)$, in appropriate senses, with the intention that that the solution U_h approximates u.

3.2. **Abstract stability estimate.** We first wish to show that there exists a solution to our discrete scheme satisfying a stability bound similar to (2.5) for the continuous case. We start by making assumptions on the bilinear forms m_h and a_h. We assume that all constants are independent of $h \in (0,h_0)$.

Assumptions on m_h: We assume that $m_h(t;\cdot,\cdot)$ is symmetric:

$$m_h(t;V_h,\phi_h) = m_h(t;\phi_h,V_h) \qquad \text{for } V_h,\phi_h \in \mathcal{V}_h(t). \tag{M$_h$1}$$

We assume that there exists $c_1, c_2 > 0$ such that for all $t \in [0,T]$, we have

$$c_1 \|V_h\|_{\mathcal{H}_h(t)} \le \big(m_h(t;V_h,V_h)\big)^{\frac{1}{2}} \le c_2 \|V_h\|_{\mathcal{H}_h(t)} \qquad \text{for } V_h \in \mathcal{V}_h(t). \tag{M$_h$2}$$

We assume the existence of a bilinear form $g_h(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t) \to \mathbb{R}$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;V_h,V_h) = m_h(t;\partial_h^\bullet V_h,V_h) + \frac{1}{2} g_h(t;V_h,V_h) \qquad \text{for } V_h \in C^1_{\mathcal{V}_h}. \tag{G$_h$1}$$

We assume that there exists $c_3 > 0$ such that for all $t \in [0,T]$

$$|g_h(t;V_h,\phi_h)| \le c_3 \|V_h\|_{\mathcal{H}_h(t)} \|\phi_h\|_{\mathcal{H}_h(t)} \qquad \text{for } V_h,\phi_h \in \mathcal{V}_h(t). \tag{G$_h$2}$$

Assumptions on a_h: We assume that the map

$$t \mapsto a_h(t;V_h,\phi_h) \qquad \text{for } V_h,\phi_h \in L^2_{\mathcal{V}_h}, \tag{A$_h$1}$$

is measurable, and there exists constants $c_4, c_5, c_6 > 0$ such that for all $t \in [0,T]$, we have

$$a_h(t;V_h,V_h) \ge c_4 \|V_h\|^2_{\mathcal{V}_h(t)} - c_5 \|V_h\|^2_{\mathcal{H}_h(t)} \qquad \text{for } V_h \in \mathcal{V}_h(t) \tag{A$_h$2}$$

$$|a_h(t;V_h,\phi_h)| \le c_6 \|V_h\|_{\mathcal{V}_h(t)} \|\phi_h\|_{\mathcal{V}_h(t)} \qquad \text{for } V_h,\phi_h \in \mathcal{V}_h(t). \tag{A$_h$3}$$

Finally, we assume the existence of a bilinear form $b_h(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t) \to \mathbb{R}$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_h(t;V_h,V_h) = a_h(t;\partial_h^\bullet V_h,V_h) + \frac{1}{2} b_h(t;V_h,V_h) \qquad \text{for } V_h \in C^1_{\mathcal{V}_h}. \tag{B$_h$1}$$

We assume that there exists $c_8 > 0$ such that for all $t \in [0,T]$ we have

(B$_h$2) $$|b_h(t;V_h,\phi_h)| \leq c_8 \|V_h\|_{\mathcal{V}_h(t)} \|\phi_h\|_{\mathcal{V}_h(t)} \qquad \text{for } V_h, \phi_h \in \mathcal{V}_h(t).$$

We note that due to Assumption (G$_h$1), the finite element scheme (3.2) can be re-written as

(3.5) $$m_h(t;\partial_h^\bullet U_h,\phi_h) + g_h(t;U_h,\phi_h) + a_h(t;U_h,\phi_h) = 0 \qquad \text{for } \phi_h \in C^1_{\mathcal{V}_h}.$$

Theorem 3.3 (Existence and stability of finite element method)**.** *Let Assumptions* (M$_h$1)*,* (M$_h$2)*,* (G$_h$1)*,* (G$_h$2)*,* (A$_h$1)*,* (A$_h$2)*,* (A$_h$3)*,* (B$_h$1) *and* (B$_h$2) *hold with constants independent of $h \in (0,h_0)$. Then* (3.2) *has a unique solution $U_h \in C^0_{\mathcal{V}_h}$ with $\partial_h^\bullet U_h \in C^0_{\mathcal{V}_h}$. There exists a constant $C > 0$ independent of $h \in (0,h_0)$ such that*

(3.6) $$\sup_{t\in(0,T)} \|U_h\|^2_{\mathcal{H}_h(t)} + \int_0^T \|U_h\|^2_{\mathcal{V}_h(t)}\,\mathrm{d}t \leq C \left\|U_{h,0}\right\|^2_{\mathcal{H}_h(t)}.$$

Proof. We consider the problem in the matrix from (3.4). Since $\mathcal{M}(\cdot) \in C^1(0,T;R^{N\times N})$ (G$_h$1) and is invertible (M$_h$2), this is equivalent to

(3.7) $$\alpha'(t) + \mathcal{M}^{-1}(t)\big(\mathcal{M}'(t) + \mathcal{S}(t)\big)\alpha(t) = 0.$$

This is a linear autonomous system of ordinary equations with C^0 coefficients (easily verified). Standard theory implies there exists a unique solution $\alpha \in C^1(0,T;\mathbb{R}^N)$, which can be translated as $U_h \in C^1_{\mathcal{V}_h}$.

To show the energy bound, we start by testing (3.2) with $\phi_h = U_h$:

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;U_h,U_h) + a_h(t;U_h,U_h) - m_h(t;U_h,\partial_h^\bullet U_h) = 0.$$

The transport inequality implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;U_h,U_h) - m_h(t;U_h,\partial_h^\bullet U_h) = \frac{1}{2} g_h(t;U_h,U_h),$$

thus we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;U_h,U_h) + a_h(t;U_h,U_h) \leq -\frac{1}{2} g_h(t;U_h,U_h).$$

Using the bounds from (M$_h$2), (A$_h$3) and (G$_h$2) and integrating in time gives

$$\frac{c_1}{2} \|U_h\|^2_{\mathcal{H}_h(t)} + c_4 \int_0^T \|U_h\|^2_{\mathcal{V}_h(t)}\,\mathrm{d}t \leq \left(c_5 + \frac{c_3}{2}\right) \int_0^T \|U_h\|^2_{\mathcal{H}_h(t)}\,\mathrm{d}t + \frac{c_2}{2}\left\|U_{h,0}\right\|^2_{\mathcal{H}_h(t)}.$$

Applying a Gronwall inequality gives the desired bound. □

3.3. **Abstract error analysis.** In order to prove an error estimate, we make two sets of assumptions. The first concerns the smoothness of continuous problem and a related dual problem and the second relates the discrete structures with their continuous counterparts. We assume the following constants are independent of $h \in (0,h_0)$.

We require two further time dependent Hilbert spaces $\{\mathcal{Z}_0(t)\}$ and $\{\mathcal{Z}(t)\}$ which satisfy $\mathcal{Z}(t) \subset \mathcal{Z}_0(t) \subset \mathcal{V}(t)$ for each $t \in [0,T]$ with the embeddings uniformly continuous. These spaces represent spaces of smooth functions.

3.3.1. *Approximation assumptions.* In this section we list the assumptions required to show an error bound in Theorem 3.8. The below assumptions are in addition to the assumptions of Theorems 2.9 and 3.3. We fix $k \geq 1$.

First, we place a further requirement on the bilinear form b. We assume that there exists a constant $c_9 > 0$ such that for all $t \in [0,T]$ and all $\eta \in \mathcal{V}(t)$ and $\varphi \in \mathcal{Z}_0(t)$ we have

$$|b(t;\eta,\varphi)| \leq c_9 \|\eta\|_{\mathcal{H}(t)} \|\varphi\|_{\mathcal{Z}_0(t)}. \tag{B3}$$

Lifted space assumptions: For each $t \in [0,T]$, we assume that there exists a lifting map $\Lambda_h(\cdot,t)\colon \mathcal{V}_h(t) \to \mathcal{V}(t)$ such that there exists $c_1, c_2 > 0$ such that for all $t \in [0,T]$ and all $V_h \in \mathcal{V}_h(t)$ with lift $v_h = V_h^\ell$

$$c_1 \|v_h\|_{\mathcal{H}(t)} \leq \|V_h\|_{\mathcal{H}_h(t)} \leq c_2 \|v_h\|_{\mathcal{H}(t)} \tag{L1}$$

$$c_1 \|v_h\|_{\mathcal{V}(t)} \leq \|V_h\|_{\mathcal{V}_h(t)} \leq c_2 \|v_h\|_{\mathcal{V}(t)}. \tag{L2}$$

We denote by $\mathcal{V}_h^\ell(t) \subset \mathcal{V}_h(t)$ the image of $\mathcal{V}_h(t)$ under the map $\Lambda_h(\cdot,t)$. Let $\widetilde{\phi}_t^\ell\colon \mathcal{V}_h^\ell(0) \to \mathcal{V}_h^\ell(t)$ be given by

$$\widetilde{\phi}_t^\ell(\Lambda_h(V_h,0)) := \Lambda_h(\phi_t^h(V_h),t).$$

Our assumptions imply that the pair $\{\mathcal{V}_h^\ell(t), \widetilde{\phi}_t^\ell\}_{t\in[0,T]}$ is compatible (uniformly in h) in the $\mathcal{H}(t)$ and $\mathcal{V}(t)$-norms.

We will assume further that there exists a map $\phi_t^\ell\colon \mathcal{H}_0 \to \mathcal{H}(t)$ such that $\phi_t^\ell|_{\mathcal{V}_h(0)} = \widetilde{\phi}_t^\ell$ and that $\{\mathcal{H}(t), \phi_t^\ell\}_{t\in[0,T]}$ and $\{\mathcal{V}(t), \phi_t^\ell\}_{t\in[0,T]}$ are compatible pairs (again uniformly for $h \in (0,h_0)$). We denote by $\partial_h^\bullet v_h$ the material derivative for the push-forward map ϕ_t^ℓ for $v_h \in C^1_{\mathcal{V}_h^\ell}$.

We assume that we have a transport formula for functions in $C^1_{\mathcal{H}}$ and $C^1_{\mathcal{V}}$ for the push-forward map ϕ_t^ℓ for the m and a bilinear forms. We assume that there exists bilinear forms $\widetilde{g}_h(t;\cdot,\cdot)\colon \mathcal{H}(t)\times\mathcal{H}(t) \to \mathbb{R}$ and $\widetilde{b}_h(t;\cdot,\cdot)\colon \mathcal{V}(t)\times\mathcal{V}(t) \to \mathbb{R}$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m(t;\eta,\eta) = m(t;\partial_h^\bullet\eta,\eta) + \frac{1}{2}\widetilde{g}_h(t;\eta,\eta) \qquad \text{for } \eta \in C^1_{\mathcal{H}} \tag{$\widetilde{\text{G}}$1}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a(t;\eta,\eta) = a(t;\partial_h^\bullet\eta,\eta) + \frac{1}{2}\widetilde{b}_h(t;\eta,\eta) \qquad \text{for } \eta \in C^1_{\mathcal{V}}. \tag{$\widetilde{\text{B}}$1}$$

Note that these bilinear forms $\widetilde{g}_h$ and $\widetilde{b}_h$ may depend on $h \in (0,h_0)$. We assume that for these bilinear forms there exists $c_1, c_2 > 0$ such that for all $t \in [0,T]$ and all $h \in (0,h_0)$,

$$|\widetilde{g}_h(t;\eta,\varphi)| \leq c_1 \|\eta\|_{\mathcal{H}(t)} \|\varphi\|_{\mathcal{H}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{H}(t) \tag{$\widetilde{\text{G}}$2}$$

$$\left|\widetilde{b}_h(t;\eta,\varphi)\right| \leq c_2 \|\eta\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{V}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{V}(t). \tag{$\widetilde{\text{B}}$2}$$

For each time $t \in [0,T]$, we will also use an inverse lift $\mathcal{E}_h(\cdot,t)\colon \mathcal{Z}_0(t) \to \mathcal{Z}_0^{-\ell}(t)$ and use the notation $\mathcal{E}_h(\eta,t) = \eta^{-\ell}$. We assume that there exists $c_1, c_2 > 0$ such that for all $t \in [0,T]$ and all $\eta \in \mathcal{Z}_0(t)$ with inverse lift $\eta^{-\ell} \in \mathcal{Z}_0^{-\ell}(t)$

$$c_1 \|\eta\|_{\mathcal{H}(t)} \leq \left\|\eta^{-\ell}\right\|_{\mathcal{H}_h(t)} \leq c_2 \|\eta\|_{\mathcal{H}(t)} \tag{L3}$$

$$c_1 \|\eta\|_{\mathcal{V}(t)} \leq \left\|\eta^{-\ell}\right\|_{\mathcal{V}_h(t)} \leq c_2 \|\eta\|_{\mathcal{V}(t)}. \tag{L4}$$

Approximation property of $\mathcal{V}_h^\ell(t)$: For each $t \in [0,T]$, we assume that there exists a well defined interpolation operator $I_h\colon \mathcal{Z}_0(t) \to \mathcal{V}_h^\ell(t)$ such that there exists a constant $c > 0$

such that for all $t \in [0,T]$

(I1) $\quad \|\eta - I_h\eta\|_{\mathcal{H}(t)} + h\|\eta - I_h\eta\|_{\mathcal{V}(t)} \leq ch^2 \|\eta\|_{\mathcal{Z}_0(t)} \qquad$ for $\eta \in \mathcal{Z}_0(t)$

(I2) $\quad \|\eta - I_h\eta\|_{\mathcal{H}(t)} + h\|\eta - I_h\eta\|_{\mathcal{V}(t)} \leq ch^{k+1} \|\eta\|_{\mathcal{Z}(t)} \qquad$ for $\eta \in \mathcal{Z}(t)$.

Assumptions on the geometric approximation: Finally, we assume we have the following relations between continuous and discrete bilinear forms. We assume that there exists constants $c > 0$ such that for all $t \in [0,T]$ the following holds for all $V_h, \phi_h \in \mathcal{V}_h(t) + \mathcal{Z}_0^{-\ell}(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t) + \mathcal{Z}_0(t)$ we have

(P1) $\quad |m(t; v_h, \varphi_h) - m_h(t; V_h, \phi_h)| \leq ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$

(P2) $\quad |\widetilde{g}_h(t; v_h, \varphi_h) - g_h(t; V_h, \phi_h)| \leq ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$

(P3) $\quad |a(t; v_h, \varphi_h) - a_h(t; V_h, \phi_h)| \leq ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$

(P4) $\quad \left|\widetilde{b}_h(t; v_h, \varphi_h) - b_h(t; V_h, \phi_h)\right| \leq ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$

(P5) $\quad \left|\widetilde{b}_h(t; v_h, \varphi_h) - b(t; v_h, \varphi_h)\right| \leq ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$.

For $\eta, \varphi \in \mathcal{Z}_0(t)$ with inverse lifts $\eta^{-\ell}, \varphi^{-\ell}$, we have

(P3') $\quad \left|a(t; \eta, \varphi) - a_h(t; \eta^{-\ell}, \varphi^{-\ell})\right| \leq ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)}$

(P4') $\quad \left|\widetilde{b}_h(t; \eta, \varphi) - b_h(t; \eta^{-\ell}, \varphi^{-\ell})\right| \leq ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)}$

(P6) $\quad \left|a(t; \partial_h^\bullet \eta, \varphi) - a_h(t; \partial_h^\bullet \eta^{-\ell}, \varphi^{-\ell})\right| \leq ch^{k+1} (\|\eta\|_{\mathcal{Z}_0(t)} + \|\partial^\bullet \eta\|_{\mathcal{Z}_0(t)}) \|\varphi\|_{\mathcal{Z}_0(t)}$

Finally, we assume

(P7) $\quad \|\partial_h^\bullet \varphi - \partial^\bullet \varphi\|_{\mathcal{H}(t)} \leq ch^{k+1} \|\varphi\|_{\mathcal{V}(t)} \qquad$ for $\varphi \in \mathcal{V}(t)$

(P8) $\quad \|\partial_h^\bullet \varphi - \partial^\bullet \varphi\|_{\mathcal{V}(t)} \leq ch^{k} \|\varphi\|_{\mathcal{Z}_0(t)} \qquad$ for $\varphi \in \mathcal{Z}_0(t)$.

Assumptions on regularity of a dual problem: Let $\kappa > 0$ be such that $a(t;\cdot,\cdot) + \kappa m(t;\cdot,\cdot)$ is positive definite. We introduce the dual problem: Given $\xi \in \mathcal{H}(t)$, find $\zeta \in \mathcal{V}(t)$ such that

(3.8) $\quad a(t; \chi, \zeta) + \kappa m(t; \chi, \zeta) = m(t; \xi, \chi) \qquad$ for $\chi \in \mathcal{V}(t)$.

Assumptions on κ along with the previous assumptions imply that (3.8) has a unique solution and we assume the regularity condition that there exists $c > 0$ such that

(R2) $\quad \|\zeta\|_{\mathcal{Z}_0(t)} \leq c \|\xi\|_{\mathcal{H}(t)}$,

where the constant is independent of ξ and time t.

3.3.2. *Ritz projection.* We will introduce a Ritz projection in the following with respect to modified positive definite bilinear forms a^κ and a_h^κ. We know from Assumptions (A2) and (A_h2), there exists $\kappa > 0$ such that there exists $c > 0$ such that for all $t \in [0,T]$ and all $h \in (0, h_0)$

(3.9) $\quad a^\kappa(t; \eta, \eta) = a(t; \eta, \eta) + \kappa m(t; \eta, \eta) \geq c \|\eta\|_{\mathcal{V}(t)}^2 \qquad$ for $\eta \in \mathcal{V}(t)$

(3.10) $\quad a_h^\kappa(t; V_h, V_h) = a_h(t; V_h, V_h) + \kappa m_h(t; V_h, V_h) \geq c \|V_h\|_{\mathcal{V}_h(t)}^2 \qquad$ for $V_h \in \mathcal{V}_h(t)$.

We suppose now that κ is fixed in the sequel. We continue by quoting results which follow by combining properties of a and m or a_h and m_h. From (A3) and (M2), or (A$_h$3) and (M$_h$2), there exists a constant $c > 0$ such that

$$|a^\kappa(t;\eta,\varphi)| \le c\,\|\eta\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{V}(t) \tag{3.11}$$

$$|a_h^\kappa(t;v_h,\phi_h)| \le c\,\|v_h\|_{\mathcal{V}_h(t)}\,\|\phi_h\|_{\mathcal{V}_h(t)} \qquad \text{for } v_h,\phi_h \in \mathcal{V}_h(t). \tag{3.12}$$

We assume further that a^κ is differentiable in time so that there exists a bilinear forms $b^\kappa(t;\cdot,\cdot)$ and $\widetilde{b}_h^\kappa(t;\cdot,\cdot)$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a^\kappa(t;\eta,\varphi) = a^\kappa(t;\partial^\bullet\eta,\varphi) + a^\kappa(t;\eta,\partial^\bullet\varphi) + b^\kappa(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{V}} \tag{3.13}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a^\kappa(t;\eta,\varphi) = a^\kappa(t;\partial_h^\bullet\eta,\varphi) + a^\kappa(t;\eta,\partial_h^\bullet\varphi) + \widetilde{b}_h^\kappa(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{V}}. \tag{3.14}$$

We assume that the bilinear forms b^κ and $\widetilde{b}_h^\kappa$ are uniformly bounded in the following senses, there exists a constant $c > 0$ such that for all $t \in [0,T]$

$$|b^\kappa(t;\eta,\varphi)| \le c\,\|\eta\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{V}(t) \tag{3.15}$$

$$|b^\kappa(t;\eta,\varphi)| \le c\,\|\eta\|_{\mathcal{H}(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)} \qquad \text{for } \eta \in \mathcal{V}(t), \varphi \in \mathcal{Z}_0(t) \tag{3.16}$$

$$\left|\widetilde{b}_h^\kappa(t;\eta,\varphi)\right| \le c\,\|\eta\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for } \eta,\varphi \in \mathcal{V}(t). \tag{3.17}$$

We can combine transport formula for (G$_h$1) and (B$_h$1) to see that there exists a bilinear form $b_h^\kappa(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t) \to \mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}a_h^\kappa(t;v_h,\phi_h) = a_h^\kappa(t;\partial_h^\bullet v_h,\phi_h) + a_h^\kappa(t;v_h,\partial_h^\bullet\phi_h) + b_h^\kappa(t;v_h,\phi_h) \qquad \text{for } v_h,\phi_h \in \mathcal{V}_h(t). \tag{3.18}$$

Further, we know that b_h^κ is bounded: There exists a constant $c > 0$ such that

$$|b_h^\kappa(t;v_h,\phi_h)| \le c\,\|v_h\|_{\mathcal{V}_h(t)}\,\|\phi_h\|_{\mathcal{V}_h(t)}. \tag{3.19}$$

Finally, we note that the following estimates hold for $V_h,\phi_h \in \mathcal{V}_h(t) + \mathcal{Z}_0^{-\ell}(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t) + \mathcal{Z}_0(t)$:

$$|a^\kappa(t;v_h,\varphi_h) - a_h^\kappa(t;V_h,\phi_h)| \le ch^k\,\|v_h\|_{\mathcal{V}(t)}\,\|\varphi_h\|_{\mathcal{V}_h(t)} \tag{3.20}$$

$$\left|\widetilde{b}_h^\kappa(t;v_h,\varphi_h) - b_h^\kappa(t;V_h,\phi_h)\right| \le ch^k\,\|v_h\|_{\mathcal{V}(t)}\,\|\varphi_h\|_{\mathcal{V}_h(t)} \tag{3.21}$$

$$\left|b^\kappa(t;v_h,\varphi_h) - \widetilde{b}_h^\kappa(t;v_h,\varphi_h)\right| \le ch^k\,\|v_h\|_{\mathcal{V}(t)}\,\|\varphi_h\|_{\mathcal{V}_h(t)}. \tag{3.22}$$

Furthermore, for $\eta,\varphi \in \mathcal{Z}_0(t)$, we have

$$\left|a^\kappa(t;\eta,\varphi) - a_h^\kappa(t;\eta^{-\ell},\varphi^{-\ell})\right| \le ch^{k+1}\,\|\eta\|_{\mathcal{Z}_0(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)} \tag{3.23}$$

$$\left|b^\kappa(t;\eta,\varphi) - b_h^\kappa(t;\eta^{-\ell},\varphi^{-\ell})\right| \le ch^{k+1}\,\|\eta\|_{\mathcal{Z}_0(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)}. \tag{3.24}$$

Definition 3.4. The Ritz projection as an operator $\Pi_h\colon \mathcal{V}(t) \to \mathcal{V}_h(t)$. For $z \in \mathcal{V}(t)$, $\Pi_h z$ is given as the unique solution of

$$a_h^\kappa(t;\Pi_h z,\phi_h) = a^\kappa(t;z,\varphi_h) \qquad \text{for all } \phi_h \in \mathcal{V}_h(t) \text{ with lift } \varphi_h = \phi_h^\ell. \tag{3.25}$$

We denote by $\pi_h z = (\Pi_h z)^\ell \in \mathcal{V}_h^\ell(t)$.

Remark 3.5. By including the domain perturbation in the Ritz projection, we can apply the Narrow band trace inequality for optimal error bounds for bulk equations in the applications we consider. A similar approach is used by Du, Ju, and Tian (2011) and Elliott and Ranner (2014) in the context of Cahn-Hilliard equations on stationary and evolving surfaces.

Lemma 3.6. *For each $z \in \mathcal{V}(t)$, there exists a unique solution $\Pi_h z$ of (3.25). There exists a constant $c > 0$ such that for all $h \in (0,h_0)$ and all $t \in [0,T]$ we have*

$$\|\Pi_h z\|_{\mathcal{V}_h(t)} \leq c\,\|z\|_{\mathcal{V}(t)} \qquad \text{for } z \in \mathcal{V}(t). \tag{3.26}$$

Furthermore, if $z \in \mathcal{Z}(t)$, we have the estimate

$$\|z-\pi_h z\|_{\mathcal{H}(t)} + h\,\|z-\pi_h z\|_{\mathcal{V}(t)} \leq ch^{k+1}\,\|z\|_{\mathcal{Z}(t)}\,, \tag{3.27}$$

for a constant $c > 0$ independent of $t \in [0,T]$ and $h \in (0,h_0)$.

Proof. Since a_h^κ is uniformly coercive (3.10) and bounded (3.12) and a^κ is bounded (3.11), it is clear that there exists a unique solution that satisfies the stability bound (3.26).

To show the error bound, we consider the functional $F_h \colon \mathcal{V}(t) \to \mathbb{R}$ given by

$$F_h(\varphi) = a^\kappa(t; z-\pi_h z, \varphi).$$

First, note that for $\varphi = \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t)$, we can use the definition of $\Pi_h z$ (3.25) to see that

$$F_h(\varphi_h) = a^\kappa(t; z-\pi_h z, \varphi_h) = a_h^\kappa(t;\Pi_h z, \phi_h) - a^\kappa(t;\pi_h z, \varphi_h).$$

Then the perturbation estimate (3.20) and the stability bound (3.26) imply that

$$|F_h(\varphi_h)| \leq ch^k\,\|\Pi_h z\|_{\mathcal{V}_h(t)}\,\|\varphi_h\|_{\mathcal{V}(t)} \leq ch^k\,\|z\|_{\mathcal{V}(t)}\,\|\varphi_h\|_{\mathcal{V}(t)}\,. \tag{3.28}$$

Next, we consider $F_h(\varphi)$ for $\varphi \in \mathcal{Z}_0$. Then, again using (3.25) we have

$$\begin{aligned}
F_h(\varphi) &= a^\kappa(t; z-\pi_h z, \varphi) \\
&= a^\kappa(t; z-\pi_h z, \varphi - I_h\varphi) + a^\kappa(t; z-\pi_h z, I_h\varphi) \\
&= a^\kappa(t; z-\pi_h z, \varphi - I_h\varphi) + \big(a_h^\kappa(t;\Pi_h z, (I_h\varphi)^{-\ell}) - a^\kappa(t;\pi_h z, I_h\varphi)\big) =: I_1 + I_2.
\end{aligned}$$

Using the boundedness of a^κ (3.11) and the interpolation bounds (I1), we have

$$|I_1| \leq c\,\|z-\pi_h\|_{\mathcal{V}(t)}\,\|\varphi - I_h\varphi\|_{\mathcal{V}(t)} \leq ch\,\|z-\pi_h\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)}\,.$$

We split I_2 so that together with the perturbation estimates (3.20) and (3.23) and the interpolation result we have

$$\begin{aligned}
|I_2| \leq{}& \Big|a_h^\kappa(t;\Pi_h z, (I_h\varphi - \varphi)^{-\ell}) - a^\kappa(t;\pi_h z, I_h\varphi - \varphi)\Big| \\
&+ \Big|a_h^\kappa(t;\Pi_h z - z^{-\ell}, (\varphi)^{-\ell}) - a^\kappa(t;\pi_h z - z, \varphi)\Big| \\
&+ \Big|a_h^\kappa(t; z^{-\ell}, (\varphi)^{-\ell}) - a^\kappa(t; z, \varphi)\Big| \\
\leq{}& ch^{2k}\,\|\Pi_h z\|_{\mathcal{V}_h(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)} + ch^k\,\|\pi_h z - z\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \\
&+ ch^{k+1}\,\|z\|_{\mathcal{Z}_0(t)}\,\|\varphi\|_{\mathcal{Z}_0(t)}\,.
\end{aligned}$$

Then combining the above estimates with the stability bound (3.26), we see that

$$|F_h(\varphi)| \leq c\big(h\,\|z-\pi_h z\|_{\mathcal{V}(t)} + h^{k+1}\,\|z\|_{\mathcal{Z}(t)}\big)\,\|\varphi\|_{\mathcal{Z}_0(t)}\,. \tag{3.29}$$

To show the $\mathcal{V}(t)$-norm error bound, we have

$$a^\kappa(t; z-\pi_h z, z-\pi_h z) = a^\kappa(t; z-\pi_h z, z - I_h z) + F_h(I_h z - \pi_h z).$$

Applying the boundedness and coercivity of a^κ (3.11) and (3.9), the interpolation bound (I2) and the first bound on F_h (3.28) gives

$$\|z - \pi_h z\|_{\mathcal{V}(t)}^2 \le ch \|z - \pi_h z\|_{\mathcal{V}(t)} \|z\|_{\mathcal{Z}(t)} + ch^k \|z\|_{\mathcal{V}(t)} \|I_h z - \pi_h z\|_{\mathcal{V}(t)} .$$

Reintroducing z into the final term, then using the interpolation bound (I2) and rearranging using a Young's inequality gives

$$\|z - \pi_h z\|_{\mathcal{V}(t)} \le ch^k \|z\|_{\mathcal{Z}(t)} . \tag{3.30}$$

For the $\mathcal{H}(t)$-norm bound, we consider the dual problem (3.8) with $\xi = z - \pi_h z \in \mathcal{H}(t)$. Then there exists a unique $\zeta \in \mathcal{V}(t)$ such that

$$a^\kappa(t; \varphi, \zeta) = m(t; \xi, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}(t).$$

Furthermore, $\zeta \in \mathcal{Z}_0(t)$ and satisfies (R2)

$$\|\zeta\|_{\mathcal{Z}_0(t)} \le c \|z - \pi_h z\|_{\mathcal{H}(t)} . \tag{3.31}$$

Then we have from (M2) that

$$\|z - \pi_h z\|_{\mathcal{H}(t)}^2 \le c_2 m(t; z - \pi_h z, z - \pi_h z) = c_2 a^\kappa(t; z - \pi_h z, \zeta) = c_2 F_h(\zeta).$$

Then the second bound on F_h (3.29) together with the $\mathcal{V}(t)$-norm bound (3.30) and the dual regularity estimate (3.31) imply that

$$\begin{aligned} \|z - \pi_h z\|_{\mathcal{H}(t)}^2 &\le \big(ch \|z - \pi_h z\|_{\mathcal{V}(t)} + ch^{k+1} \|z\|_{\mathcal{Z}(t)} \big) \|\zeta\|_{\mathcal{Z}_0(t)} \\ &\le ch^{k+1} \|z\|_{\mathcal{Z}(t)} \|z - \pi_h z\|_{\mathcal{H}(t)} . \end{aligned}$$

Rearranging this bound provides the $\mathcal{H}(t)$-norm bound. □

Since in general the material derivative and Ritz projection do not commute, we must provide a further estimate for this material derivative of the error $z - \pi_h z$. First we notice that we can take a time derivative of (3.25) and use ($B_h 1$) and (3.13) or (3.14) to see that $\partial_h^\bullet \Pi_h z$ satisfies

$$a_h^\kappa(t; \partial_h^\bullet \Pi_h z, \phi_h) = a^\kappa(t; \partial^\bullet z, \varphi_h) + b^\kappa(t; z, \varphi_h) - b_h^\kappa(t; \Pi_h z, \phi_h) \tag{3.32}$$

$$a_h^\kappa(t; \partial_h^\bullet \Pi_h z, \phi_h) = a^\kappa(t; \partial_h^\bullet z, \varphi_h) + \widetilde{b}_h^\kappa(t; z, \varphi_h) - b_h^\kappa(t; \Pi_h z, \phi_h) \tag{3.33}$$

$$\text{for all } \phi_h \in \mathcal{V}_h(t) \text{ with lift } \varphi_h = \phi_h^\ell .$$

Lemma 3.7. *We have there exists a constants $c > 0$ such that for all $t \in [0,T]$, $h \in (0,h_0)$ for $z \in \mathcal{Z}(t)$ with $\partial^\bullet z \in \mathcal{Z}(t)$*

$$\|\partial_h^\bullet \Pi_h z\|_{\mathcal{V}_h(t)} \le c \big(\|z\|_{\mathcal{V}(t)} + \|\partial^\bullet z\|_{\mathcal{V}(t)} \big) \tag{3.34}$$

and

$$\|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{H}(t)} + h \|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{V}(t)} \le ch^{k+1} \big(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \big). \tag{3.35}$$

Proof. For the stability bound, we see that $\partial_h^\bullet \Pi_h z$ satisfies the discrete elliptic problem (3.32). This tells us that $\partial_h^\bullet \Pi_h z \in \mathcal{V}_h(t)$ and, combined with the boundedness of a^κ (3.11), b^κ (3.15) and b_h^κ (3.19) and the stability estimate (3.26), satisfies the stability bound presented in (3.34).

To show the error bound, we proceed in a similar fashion to Lemma 3.6, we introduce the functional $T_h \colon \mathcal{V}(t) \to \mathbb{R}$ given by

$$T_h(\varphi) = a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \varphi).$$

First, for $\varphi = \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t)$, we can use (3.33) to see

$$\begin{aligned} T_h(\varphi_h) &= a^\kappa(t; \partial_h^\bullet(z - \pi_h z), \varphi_h) \\ &= \big(a_h^\kappa(t; \partial_h^\bullet \Pi_h z, \phi_h) - a^\kappa(t; \partial_h^\bullet \pi_h z, \varphi_h)\big) + \big(b_h^\kappa(t; \Pi_h z, \phi_h) - \widetilde{b}_h^\kappa(t; \pi_h z, \varphi_h)\big) \\ &\quad + \widetilde{b}_h^\kappa(t; \Pi_h z - z, \varphi_h). \end{aligned}$$

Then, using the perturbation estimates on a^κ (3.20) and b^κ (3.21), the boundedness of $\widetilde{b}_h^\kappa$ (3.17), the error bound (3.27) and the stability estimates (3.26) and (3.34) gives

$$\begin{aligned} |T_h(\varphi_h)| &\le ch^k \big(\|\Pi_h z\|_{\mathcal{V}_h(t)} + \|\partial_h^\bullet \Pi_h z\|_{\mathcal{V}_h(t)} + \|z\|_{\mathcal{Z}(t)} \big) \|\varphi_h\|_{\mathcal{V}(t)} \\ &\le ch^k \big(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \big) \|\varphi_h\|_{\mathcal{V}(t)}. \end{aligned} \tag{3.36}$$

Secondly, for $\varphi \in \mathcal{Z}_0(t)$, we have using (3.33)

$$\begin{aligned} T_h(\varphi) &= a^\kappa(t; \partial_h^\bullet(z - \pi_h z), \varphi - I_h\varphi) + a^\kappa(t; \partial_h^\bullet(z - \pi_h z), I_h\varphi) \\ &= a^\kappa(t; \partial_h^\bullet(z - \pi_h z), \varphi - I_h\varphi) \\ &\quad + \big(a_h^\kappa(t; \partial_h^\bullet \Pi_h z, (I_h\varphi)^{-\ell}) - a_h^\kappa(t; \partial_h^\bullet \pi_h z, I_h\varphi)\big) \\ &\quad + \big(b_h^\kappa(t; \Pi_h z, (I_h\varphi)^{-\ell}) - \widetilde{b}_h^\kappa(t; \pi_h z, I_h\varphi)\big) \\ &\quad + \widetilde{b}_h^\kappa(t; \pi_h z - z, I_h\varphi) \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We split the four terms $I_1, \dots, I_4$ using the smooth functions z and φ so that

$$\begin{aligned} I_1 &= a^\kappa(t; \partial_h^\bullet(z - \pi_h z), \varphi - I_h\varphi) \\ I_2 &= \big(a_h^\kappa(t; \partial_h^\bullet \Pi_h z, (I_h\varphi - \varphi)^{-\ell}) - a^\kappa(t; \partial_h^\bullet \pi_h z, I_h\varphi - \varphi)\big) \\ &\quad + \big(a_h^\kappa(t; \partial_h^\bullet(\Pi_h z - z^{-\ell}), (\varphi)^{-\ell}) - a^\kappa(t; \partial_h^\bullet(\pi_h z - z), \varphi)\big) \\ &\quad + \big(a_h^\kappa(t; \partial_h^\bullet(z^{-\ell}) - (\partial^\bullet z)^{-\ell}, (\varphi)^{-\ell}) - a^\kappa(t; \partial_h^\bullet z - \partial^\bullet z, \varphi)\big) \\ &\quad + \big(a_h^\kappa(t; (\partial^\bullet z)^{-\ell}, (\varphi)^{-\ell}) - a^\kappa(t; \partial^\bullet z, \varphi)\big) \\ I_3 &= \big(b_h^\kappa(t; \Pi_h z, (I_h\varphi - \varphi)^{-\ell}) - \widetilde{b}_h^\kappa(t; \pi_h z, I_h\varphi - \varphi)\big) \\ &\quad + \big(b_h^\kappa(t; \Pi_h z - z^{-\ell}, (\varphi)^{-\ell}) - \widetilde{b}_h^\kappa(t; \pi_h z - z, \varphi)\big) \\ &\quad + \big(b_h^\kappa(t; z^{-\ell}, (\varphi)^{-\ell}) - \widetilde{b}_h^\kappa(t; z, \varphi)\big) \\ I_4 &= \widetilde{b}_h^\kappa(t; \pi_h z - z, I_h\varphi - \varphi) + \big(\widetilde{b}_h^\kappa(t; \pi_h z - z, \varphi) - b^\kappa(t; \pi_h z - z, \varphi)\big) \\ &\quad + b^\kappa(t; \pi_h z - z, \varphi). \end{aligned}$$

Using the boundedness of a^κ (3.11) the interpolation estimate (I1), we have

$$|I_1| \le ch \|\partial_h^\bullet(z - \pi_h z)\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{Z}_0(t)}.$$

Using the simple and improved perturbation errors for a^κ (3.20) and (3.23), as well as the estimate with material derivatives (P6), together with the interpolation bound (I1), we have

$$\begin{aligned} |I_2| &\le ch^{k+1} \|\partial_h^\bullet \Pi_h z\|_{\mathcal{V}_h(t)} \|\varphi\|_{\mathcal{Z}_0(t)} + ch^k \|\partial_h^\bullet(z - \pi_h z)\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{V}(t)} \\ &\quad + ch^{k+1} \|z\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{V}(t)} + ch^{k+1} \|\partial^\bullet z\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \\ &\le ch^k \|\partial_h^\bullet(z - \pi_h z)\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \\ &\quad + ch^{k+1} \big(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} + \|\partial_h^\bullet \Pi_h z\|_{\mathcal{V}_h(t)} \big) \|\varphi\|_{\mathcal{Z}_0(t)}. \end{aligned}$$

Using the simple and improved perturbation estimate for b^κ (3.21) and (3.24), the interpolation result (I1), and the Ritz $\mathcal{V}(t)$-norm error bound (3.27), we have

$$\begin{aligned}|I_3| &\le ch^{k+1} \|\Pi_h z\|_{\mathcal{V}_h(t)} \|\varphi\|_{\mathcal{Z}_0(t)} + ch^{k+1} \|z\|_{\mathcal{Z}(t)} \|\varphi\|_{\mathcal{V}(t)} + ch^{k+1} \|z\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \\ &\le ch^{k+1} \left(\|z\|_{\mathcal{Z}(t)} + \|\Pi_h z\|_{\mathcal{V}_h(t)} \right) \|\varphi\|_{\mathcal{Z}_0(t)}.\end{aligned}$$

Using the boundedness of $\widetilde{b}_h^\kappa$ (3.17), the perturbation estimate (3.22), the Ritz $\mathcal{V}(t)$ and $\mathcal{H}(t)$-norm error bounds (3.27) and the boundedness of b^κ (3.16) we have

$$\begin{aligned}|I_4| &\le ch^{k+1} \|z\|_{\mathcal{Z}(t)} \|\varphi\|_{\mathcal{Z}_0(t)} + ch^{2k} \|z\|_{\mathcal{Z}(t)} \|\varphi\|_{\mathcal{V}(t)} + ch^{k+1} \|z\|_{\mathcal{Z}(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \\ &\le ch^{k+1} \|z\|_{\mathcal{Z}(t)} \|\varphi\|_{\mathcal{Z}_0(t)}.\end{aligned}$$

Combining the previous four bounds with the stability estimates for $\Pi_h z$ (3.26) and $\partial_h^\bullet \Pi_h z$ (3.34) gives

(3.37)
$$|T_h(\varphi)| \le ch^{k+1} \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right) \|\varphi\|_{\mathcal{Z}_0(t)} + ch \|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{Z}_0(t)}.$$

To show the $\mathcal{V}(t)$-norm error bound, we start with

$$\begin{aligned}&a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \partial_h^\bullet (z - \pi_h z)) \\ &\quad = a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \partial_h^\bullet z - \partial^\bullet z) + a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \partial^\bullet z - I_h \partial^\bullet z) \\ &\qquad + a^\kappa(t; \partial_h^\bullet (z - \pi_h z), I_h \partial^\bullet z - \partial_h^\bullet \pi_h z).\end{aligned}$$

The bounds on a^κ (3.11), the perturbation estimate (P8) and the first bound on T_h (3.36) gives

$$\begin{aligned}&a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \partial_h^\bullet (z - \pi_h z)) \\ &\quad \le ch^k \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right) \left(\|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{V}(t)} + \|I_h \partial^\bullet z + \partial_h^\bullet \pi_h z\|_{\mathcal{V}(t)} \right) \\ &\quad \le ch^k \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right) \big(2 \|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{V}(t)} + \|I_h \partial^\bullet z - \partial^\bullet z\|_{\mathcal{V}(t)} \\ &\qquad\qquad + \|\partial^\bullet z - \partial_h^\bullet z\|_{\mathcal{V}(t)} \big).\end{aligned}$$

Using the interpolation bound (I2), the perturbation estimate (P8) and the coercivity of a^κ (3.9) and rearranging using a Young's inequality gives

$$\|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{V}(t)} \le ch^k \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right). \tag{3.38}$$

To show the $H(t)$-norm bound, we consider the dual problem (3.8) with $\xi = e := \partial_h^\bullet (z - \pi_h z) \in \mathcal{H}(t)$. Then, there exists $\zeta \in \mathcal{V}(t)$ such that

$$a^\kappa(t; \varphi, \zeta) = m(t; e, \varphi) \quad \text{for all } \varphi \in \mathcal{V}(t).$$

Furthermore, $\zeta \in \mathcal{Z}_0(t)$ and satisfies the bound

$$\|\zeta\|_{\mathcal{Z}_0(t)} \le c \|e\|_{\mathcal{H}(t)}. \tag{3.39}$$

Then we have

$$\|\partial_h^\bullet (z - \pi_h z)\|_{\mathcal{H}(t)}^2 \le c_2 m(t; e, e) = c_2 a^\kappa(t; \partial_h^\bullet (z - \pi_h z), \zeta) = c_2 T_h(\zeta).$$

The second bound on T_h (3.37), the $\mathcal{V}(t)$-norm error bound (3.38) and the dual regularity result (3.39) give

$$\begin{aligned}\|e\|_{\mathcal{H}(t)}^2 &\le ch^{k+1} \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right) \|\zeta\|_{\mathcal{Z}_0(t)} + ch \|e\|_{\mathcal{V}(t)} \|\zeta\|_{\mathcal{Z}_0(t)} + ch^{2k+2} \|z\|_{\mathcal{Z}(t)}^2 \\ &\le ch^{k+1} \left(\|z\|_{\mathcal{Z}(t)} + \|\partial^\bullet z\|_{\mathcal{Z}(t)} \right) \|e\|_{\mathcal{H}(t)} + ch^{2k+2} \|z\|_{\mathcal{Z}(t)}^2.\end{aligned}$$

Rearranging this inequality gives the desired $\mathcal{H}(t)$-norm bound. ☐

3.3.3. *Error bound.* To show the error bound we make the following assumption on the smoothness of the continuous problem. We assume that $u \in C^1_{\mathcal{V}}$ and that there exists a constant $C > 0$ such that u satisfies that regularity estimate

$$\sup_{t\in[0,T]} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|^2_{\mathcal{Z}(t)}\, \mathrm{d}t \le C. \tag{R1}$$

Theorem 3.8. *Let all the assumptions listed in Section 3.3.1 hold. Denote by u the solution of (2.4) and by $U_h \in C^1_{\mathcal{V}_h}$ the solution of (3.2) with lift $u_h \in C^1_{\mathcal{V}_h^\ell}$. Then, there exists constant $c > 0$ such that for $h \in (0,h_0)$ we have the error estimate*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|u-u_h\|^2_{\mathcal{H}(t)} + h^2 \int_0^T \|u-u_h\|^2_{\mathcal{V}(t)}\, \mathrm{d}t \\
&\le \left\|u-u_{h,0}\right\|^2_{\mathcal{H}(t)} + ch^{2k+2}\left(\sup_{t\in[0,T]} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|^2_{\mathcal{Z}(t)}\, \mathrm{d}t\right).
\end{aligned} \tag{3.40}$$

To show the error bound, we start by rescaling both solutions. Let $\check{u} = e^{-\kappa t}u$ and $\check{U}_h = e^{-\kappa t}U_h$, which satisfy

$$m(t;\partial^\bullet \check{u},\varphi) + g(t;\check{u},\varphi) + a^\kappa(t;\check{u},\varphi) = 0 \qquad \text{for all } \varphi \in L^2_{\mathcal{V}} \tag{3.41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\check{U}_h,\phi_h) + a_h^\kappa(t;\check{U}_h\phi_h) - m_h(t;\check{U}_h,\partial_h^\bullet\phi_h) = 0 \qquad \text{for all } \phi_h \in C^1_{\mathcal{V}_h}. \tag{3.42}$$

Our assumptions imply

$$\sup_{t\in[0,T]} \|\check{u}\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet \check{u}\|^2_{\mathcal{Z}(t)} \le e^{-\kappa t} \sup_{t\in[0,T]} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|_{\mathcal{Z}(t)} + \kappa\|u\|^2_{\mathcal{Z}(t)}\, \mathrm{d}t < C. \tag{3.43}$$

We will decompose the error as:

$$\check{u}_h - \check{u} = (\check{u}_h - \pi_h\check{u}) + (\pi_h\check{u} - \check{u}) =: \theta + \rho. \tag{3.44}$$

We already have bounds on ρ from Lemma 3.6 and 3.7, thanks to assumptions (R1), so it remains to show a bound for θ. We will denote by $\vartheta = \check{U}_h - \Pi_h u$, and by our assumptions, we know $\vartheta \in C^1_{\mathcal{V}_h}$.

Lemma 3.9. *Let $\phi_h \in C^1_{\mathcal{V}_h}$ and denote by $\varphi_h = \phi_h^\ell \in C^1_{\mathcal{V}_h^\ell}$. Then ϑ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\vartheta,\phi_h) + a_h^\kappa(t;\vartheta,\phi_h) - m_h(t;\vartheta,\partial_h^\bullet\phi_h) = -E_1(\phi_h) - E_2(\phi_h), \tag{3.45}$$

where

$$\begin{aligned}
E_1(\phi_h) &= m(t;\partial_h^\bullet\rho,\varphi_h) + \widetilde{g}_h(t;\rho,\varphi_h) \\
E_2(\phi_h) &= \big(m_h(t;\partial_h^\bullet\Pi_h\check{u},\phi_h) - m(t;\partial_h^\bullet\pi_h\check{u},\varphi_h)\big) + \big(g_h(t;\Pi_h\check{u},\phi_h) - \widetilde{g}_h(t;\pi_h\check{u},\varphi_h)\big) \\
&\quad + m(t;\check{u},\partial^\bullet\varphi_h - \partial_h^\bullet\varphi_h).
\end{aligned}$$

Proof. The variational form of (3.41) and the definition of the Ritz projection (3.25) tell us that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\Pi_h\check{u},\phi_h) + a_h^\kappa(t;\Pi_h\check{u},\phi_h) - m_h(t;\Pi_h\check{u},\partial_h^\bullet\phi_h) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(m_h(t;\Pi_h\check{u},\phi_h) - m(t;\check{u},\varphi_h)\big) - \big(m_h(t;\Pi_h\check{u},\partial_h^\bullet\phi_h) - m(t;\check{u},\partial^\bullet\varphi_h)\big).
\end{aligned}$$

We use the transport formulae (G_h1) for m_h and ($\widetilde{G}1$) for m to see

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\Pi_h \check{u}, \phi_h) + a_h^\kappa(t;\Pi_h \check{u}, \phi_h) - m_h(t;\Pi_h \check{u}, \partial_h^\bullet \phi_h)\\ &= \big(m_h(t;\partial_h^\bullet \Pi_h \check{u}, \phi_h) - m(t;\partial_h^\bullet \check{u}, \varphi_h) \big) + \big(g_h(t;\Pi_h u, \phi_h) - \widetilde{g}_h(t;\check{u}, \varphi_h) \big)\\ &\quad + m(t;\check{u}, \partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h).\end{aligned}$$

Subtracting this equation from (3.42) and rearranging gives (3.45). □

Lemma 3.10. *For $\phi_h \in \mathcal{V}_h(t)$, the consistency terms E_1 and E_2 satisfy*

$$|E_1(\phi_h)| + |E_2(\phi_h)| \le ch^{k+1} \big(\|\check{u}\|_{\mathcal{Z}(t)} + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)} \big) \|\phi_h\|_{\mathcal{V}_h(t)}. \tag{3.46}$$

Proof. For E_1, we use (M2) and (2.2) together with the error bounds from (3.27) and (3.35) to see

$$\begin{aligned}|E_1(\phi_h)| &\le c\big(\|\rho\|_{\mathcal{H}(t)} + \|\partial_h^\bullet \rho\|_{\mathcal{H}(t)} \big) \|\phi_h\|_{\mathcal{H}_h}\\ &\le ch^{k+1} \big(\|\check{u}\|_{\mathcal{Z}(t)} + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)} \big) \|\phi_h\|_{\mathcal{V}_h(t)}.\end{aligned}$$

For E_2, we use the perturbation estimates (P1), (P2) and (P7) together with the stability bounds on the Ritz projection (3.26) and (3.34) to see

$$\begin{aligned}|E_2(\phi_h)| &\le ch^{k+1} \big(\|\check{u}\|_{\mathcal{H}(t)} + \|\Pi_h \check{u}\|_{\mathcal{V}_h(t)} + \|\partial_h^\bullet \Pi_h \check{u}\|_{\mathcal{V}_h(t)} \big) \|\phi_h\|_{\mathcal{V}_h(t)}\\ &\le ch^{k+1} \big(\|\check{u}\|_{\mathcal{Z}(t)} + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)} \big) \|\phi_h\|_{\mathcal{V}_h(t)}.\end{aligned}$$ □

Lemma 3.11. *The following bound holds for $\vartheta \in \mathcal{V}_h(t)$:*

$$\begin{aligned}&\sup_{t\in[0,T]} \|\vartheta\|_{\mathcal{H}_h(t)}^2 + h^2 \int_0^T \|\vartheta\|_{\mathcal{V}_h(t)}^2 \,\mathrm{d}t\\ &\qquad \le \|\vartheta\|_{\mathcal{H}_h(0)}^2 + ch^{2k+2} \int_0^T \|\check{u}\|_{\mathcal{Z}(t)}^2 + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)}^2 \,\mathrm{d}t.\end{aligned} \tag{3.47}$$

Proof. We test (3.45) with $\phi_h = \vartheta$ to see

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\vartheta,\vartheta) + a_h(t;\vartheta,\vartheta) - m_h(t;\vartheta, \partial_h^\bullet \vartheta) = -E_1(\vartheta) - E_2(\vartheta).$$

The transport formula for m_h (G_h1) tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\vartheta,\vartheta) - m_h(t;\vartheta,\partial_h^\bullet \vartheta) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\vartheta,\vartheta) + \frac{1}{2} g_h(t;\vartheta,\vartheta),$$

hence, applying the bound on E_1 and E_2 (3.46) we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;\vartheta,\vartheta) + a_h(t;\vartheta,\vartheta) = -\frac{1}{2} g_h(t;\vartheta,\vartheta) + ch^{k+1} \big(\|\check{u}\|_{\mathcal{Z}(t)} + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)} \big) \|\vartheta\|_{\mathcal{V}_h(t)}.$$

Applying the boundedness and coercivity estimates form m_h, a_h and g_h (M_h2), (A_h2) and (G_h2) with a Young's inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\vartheta\|_{\mathcal{H}_h(t)}^2 + \|\vartheta\|_{\mathcal{V}_h(t)}^2 \le c\|\vartheta\|_{\mathcal{H}_h(t)}^2 + c_\varepsilon h^{2k+2} \big(\|\check{u}\|_{\mathcal{Z}(t)}^2 + \|\partial^\bullet \check{u}\|_{\mathcal{Z}(t)}^2 \big).$$

Finally, we integrate in time using a Gröwall inequality to see the desired result. □

Finally, we can show the result of Theorem 3.8.

Proof of Theorem 3.8. We apply the splitting (3.44), the bounds on ρ from Lemma (3.6), the bounds on θ from Lemma 3.11 and the estimate on $\check{u}$ from (3.43) to see

$$\begin{aligned}
&\sup_{t\in(0,T)} \|u-u_h\|^2_{\mathcal{H}(t)} + h^2 \int_0^T \|u-u_h\|^2_{\mathcal{V}(t)}\,\mathrm{d}t \\
&\le C \sup_{t\in(0,T)} \left(\|\theta\|^2_{\mathcal{H}(t)} + \|\rho\|^2_{\mathcal{H}(t)} \right) + h^2 \int_0^T \left(\|\theta\|^2_{\mathcal{V}(t)} + \|\rho\|^2_{\mathcal{V}(t)} \right) \\
&\le Ch^{2k+2} \left(\sup_{t\in(0,T)} \|\check{u}\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet \check{u}\|^2_{\mathcal{Z}(t)}\,\mathrm{d}t \right) \\
&\le Ch^{2k+2} \left(\sup_{t\in(0,T)} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|^2_{\mathcal{Z}(t)}\,\mathrm{d}t \right). \qquad \square
\end{aligned}$$

4. Evolving domains

4.1. **Basic description of hypersurfaces.**

Definition 4.1. Let $k \in \mathbb{N} \cup \{\infty\}$. A set $\Gamma \subset \mathbb{R}^{n+1}$ is called a C^k-hypersurface if, for each point $x_0 \in \Gamma$, there exists an open set $U \subset \mathbb{R}^{n+1}$ containing x_0 and a function $\phi \in C^k(U)$ with the property that $\nabla\phi \neq 0$ on $\Gamma \cap U$ and such that

$$U \cap \Gamma = \{x \in U : \phi(x) = 0\}.$$

Given a final time $T > 0$, for each $t \in [0,T]$, we write $\Gamma(t)$ for a bounded, orientable, connected n-dimensional C^k-hypersurface ($k \ge 1$) in $\mathbb{R}^{n+1}$ and $\Gamma_0 = \Gamma(0)$. We will denote by Ω_T and $\mathscr{G}_T$ the space-time domains given by

$$\Omega_T := \bigcup_{t\in[0,T]} \Omega(t) \times \{t\}, \qquad \mathscr{G}_T := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}.$$

We introduce a signed distance function for a closed surface $\Gamma(t)$. We assume that $\Gamma(t) = \partial\Omega(t)$ is the boundary of an open bounded domain. The oriented distance function for $\Gamma(t)$ is defined by

$$d(x,t) = \begin{cases} \inf\{|x-y| : y \in \Gamma(t)\} & \text{for } x \in \mathbb{R}^{n+1} \setminus \bar{\Omega}(t) \\ -\inf\{|x-y| : y \in \Gamma(t)\} & \text{for } x \in \Omega(t) \end{cases}$$

We can orient $\Gamma(t)$ by choosing the unit normal ν as

$$\nu(x,t) = \nabla d(x,t) \qquad \text{for } x \in \Gamma(t). \tag{4.1}$$

This allows us to define the (extended) Weingarten map by $\mathcal{H} := \nabla^2 d$ and the mean curvature by $H := \operatorname{trace} \mathcal{H}$. For each time $t \in [0,T]$, there exists a narrow band $\mathcal{N}(t)$ such that the distance function $d(\cdot,t) \in C^k(\mathcal{N}(t))$ and, if $k \ge 2$, for each $x \in \mathcal{N}(t)$ there exists a unique point $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t). \tag{4.2}$$

We call the operator $p(\cdot,t)\colon \mathcal{N}(t) \to \Gamma(t)$ the normal projection operator and note that $p(\cdot,t) \in C^k(\mathcal{N}(t))$. We use this projection to extend the unit normal to be defined in $\mathcal{N}(t)$ by $\nu(x,t) = \nu(p(x,t),t)$. See (Gilbarg and Trudinger, 1983, Lemma 14.16) for details. We denote by $\mathcal{N}_T$ the space-time domain give by

$$\mathcal{N}_T := \bigcup_{t\in[0,T]} \mathcal{N}(t) \times \{t\}.$$

Given a function $\eta \colon \Gamma(t) \to \mathbb{R}$, we define its tangential gradient by

$$\nabla_\Gamma \eta := P \nabla \widetilde{\eta} \quad \text{where} \quad P_{ij}(x,t) = \delta_{ij} - \nu_i(x,t)\nu_j(x,t) \quad \text{for } x \in \Gamma(t), \tag{4.3}$$

and $\nabla \widetilde{\eta}$ is gradient of an arbitrary extension of η to $\mathcal{N}(t)$ with respect to the ambient coordinates in $\mathbb{R}^{n+1}$. It can be shown that this definition is independent of the choice of extension. The tangential gradient is a vector-valued function $\nabla_\Gamma \eta =: (\underline{D}_1 \eta, \ldots, \underline{D}_{n+1} \eta)$. This gives a natural definition of the Laplace-Beltrami operator as

$$\Delta_\Gamma \eta := \nabla_\Gamma \cdot \nabla_\Gamma \eta = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i \eta. \tag{4.4}$$

We will write integration on $\Gamma(t)$ with respect to the surface measure $\mathrm{d}\sigma$. Let $\eta \in C^1(\Gamma(t))$. The integration by parts formula is given by

$$\int_{\mathcal{M}(t)} \nabla_\Gamma \eta \,\mathrm{d}\sigma = -\int_{\mathcal{M}(t)} \eta H \nu \,\mathrm{d}\sigma + \int_{\partial \mathcal{M}(t)} \eta \mu \,\mathrm{d}\sigma \qquad \text{for } \mathcal{M}(t) \subset \Gamma(t). \tag{4.5}$$

We define $C^1_c(\Gamma(t))$ to be the space of compactly support C^1 functions on $\Gamma(t)$. If $\varphi \in C^1_c(\Gamma(t))$, then the integration by parts formula (4.5) implies

$$\int_{\Gamma(t)} \eta \underline{D}_i \varphi \,\mathrm{d}\sigma = -\int_{\Gamma(t)} \varphi \underline{D}_i \eta + \eta \varphi H \nu_i \,\mathrm{d}\sigma.$$

This identity can be used in the obvious way to define weak derivatives. We can define a family of Sobolev spaces $W^{1,q}(\Gamma(t))$ for $q \in [1,\infty]$ by

$$W^{1,q}(\Gamma(t)) = \{\eta \in L^q(\Gamma(t)) : \underline{D}_i \eta \in L^q(\Gamma(t)), i = 1, \ldots, n+1\},$$

where $\underline{D}_i \eta$ should be interpreted in the weak sense. When equipped with the usual norms the spaces $W^{1,q}(\Gamma(t))$ are Banach spaces. We will write $H^1(\Gamma(t))$ for the Hilbert space $W^{1,2}(\Gamma(t))$. One can also define higher order Sobolev spaces $W^{k,p}(\Gamma(t))$, or $H^k(\Gamma(t))$, for $k \geq 1$ if $\Gamma(t)$ is at least C^k-regular (Wloka, 1987; Hebey, 2000).

The above definitions of surface calculus are equivalent to the following given for a parametrised surface (see e.g. (Dziuk and Elliott, 2013b, page 294)). We suppose that for $x_0 \in \Gamma$, there exists an open set $U \subset \mathbb{R}^{n+1}$ with $x_0 \in U$, an open connected set $V \subset \mathbb{R}^n$ and a map $X \colon V \to U \cap \Gamma$ with the properties $X \in C^k(V, \mathbb{R}^{n+1})$, X is bijective and $\operatorname{rank} \nabla X = n$ on V. The map X is called a local parameterisation.

Let $X \in C^2(V, \mathbb{R}^{n+1})$ be a local parameterisation about a point $x_0 \in \Gamma$ and $\theta \in V$. We define the *first fundamental form* $G(\theta) = (g_{ij}(\theta))_{i,j=1,\ldots,n}$, $\theta \in V$ by

$$g_{ij}(\theta) = \frac{\partial X}{\partial \theta_i}(\theta) \cdot \frac{\partial X}{\partial \theta_j}(\theta) \qquad i, j = 1, \ldots, n.$$

We denote by (g^{ij}) the inverse of G and by $g = \det(G)$ the determinant of the matrix G. For a smooth function $f \colon \Gamma \to \mathbb{R}$, we write $F(\theta) = f(X(\theta))$, for $\theta \in V$. Then, the *tangential gradient* of f is given by

$$(\nabla_\Gamma f)(X(\theta)) = \sum_{i,j=1}^{n} g^{ij}(\theta) \frac{\partial F}{\partial \theta_j}(\theta) \frac{\partial X}{\partial \theta_i}(\theta), \tag{4.6}$$

and we can write the integral over $U \cap \Gamma$ of f as

$$\int_{U \cap \Gamma} f \,\mathrm{d}\sigma = \int_V F(\theta) \sqrt{g(\theta)} \,\mathrm{d}\theta. \tag{4.7}$$

The final result we show in this section will be useful in the error analysis of our methods.

Lemma 4.2 (Narrow band trace inequality). *For $t \in [0,T]$, let $B_\varepsilon(t) \subset \mathcal{N}(t)$ be the band given by*

$$B_\varepsilon(t) = \{x \in \Omega(t) : -\varepsilon < d(x,t) < 0\}.$$

Then there exists a constant c such that for all $t \in [0,T]$,

$$\|\eta\|_{L^2(B_\varepsilon(t))} \le c\varepsilon^{1/2} \|\eta\|_{H^1(\Omega(t))} \qquad \text{for all } \eta \in H^1(\Omega(t)). \tag{4.8}$$

Proof. The proof for stationary domains in given by Elliott and Ranner (2013, Lemma 4.10) which can be easily extended to the evolving case. □

4.2. Examples of evolving domains.

4.2.1. *Evolving closed hypersurfaces.* For each $t \in [0,T]$, let $\Gamma(t)$ be a connected, n-dimensional C^2-hypersurface with $\Gamma_0 := \Gamma(t)$. Let $\Phi_t \colon \Gamma_0 \to \Gamma(t)$ denote a flow which is of class C^2. We introduce a material velocity w defined by the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\cdot) = w(\Phi_t(\cdot),t), \qquad \Phi_0(\cdot) = \mathrm{Id}.$$

Our assumptions imply that $w(\cdot,t) \in C^2(\Gamma(t))$ uniformly in time and we assume that

$$|\nabla_\Gamma \cdot w| < C \quad \text{ for all } t \in [0,T].$$

Remark 4.3. This can be generalised to the case of an evolving sub-manifold with boundary. That is, an evolving bounded subset of a curved hypersurface whose boundary is smoothly evolving.

Remark 4.4. These assumptions are sufficient in order to describe the results on the continuous surface. Later, we will define a isoparametric finite element method that will require higher regularity of the surface and its evolution in order to derive optimal error bounds.

Let $\mathcal{V}(t) = H^1(\Gamma(t))$ and $\mathcal{H}(t) = L^2(\Gamma(t))$. We denote by $\mathcal{V}^*(t) = (H^1(\Gamma(t)))^*$. It is well known that for each time $t \in [0,T]$, $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))$ form a separable Hilbert triple. We define the push-forward operator ϕ_t by

$$(\phi_t \eta)(x,t) := \eta(\Phi_{-t}(x)) \qquad \text{for } \eta \in \mathcal{H}_0. \tag{4.9}$$

Vierling (2014, Lemma 3.4) showed that $(L^2(\Gamma(t)), \phi_t)_{t\in[0,T]}$ and $(H^1(\Gamma(t)), \phi_t)_{t\in[0,T]}$ are both compatible pairs and the spaces $L^2_{\mathcal{H}}$ and $L^2_{\mathcal{V}}$ are well defined.

For this definition of push-forward operator, we can define a strong material derivative $\partial^\bullet \eta$ of a function in $C^1_{\mathcal{H}}$ using (2.1). The transport formula (2.3) is given for $\eta, \chi \in C^1_{\mathcal{H}}$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \eta\chi \,\mathrm{d}\sigma = \int_{\Gamma(t)} \partial^\bullet\eta\chi + \chi\partial^\bullet\eta \,\mathrm{d}\sigma + g(t;\eta,\chi), \tag{4.10}$$

where

$$g(t;\eta,\chi) = \int_{\Gamma(t)} \eta\chi\nabla_\Gamma \cdot w \,\mathrm{d}\sigma.$$

More precisely, we can show that Assumption 2.4 holds for $\mathcal{H}(t) = L^2(\Gamma(t))$ (Alphonse, Elliott, and Stinner, 2015b, Section 4.1).

Furthermore, we have a transport formula for the Dirichlet inner product and advection bilinear form. Let $\mathscr{A} = \mathscr{A}(x,t)$ be a symmetric $(n+1)\times(n+1)$ diffusion tensor which is positive definite on the tangent space to $\Gamma(t)$. Then, for $\eta, \chi \in C^1_{\mathcal{V}}$, we have the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathscr{A}\nabla_\Gamma \eta \cdot \nabla_\Gamma \chi \,\mathrm{d}\sigma &= \int_{\Gamma(t)} \mathscr{A}\nabla_\Gamma \partial^\bullet \eta \cdot \nabla_\Gamma \chi + \mathscr{A}\nabla_\Gamma \eta \cdot \nabla_\Gamma \partial^\bullet \chi \,\mathrm{d}\sigma \\ &\quad + \int_{\Gamma(t)} \mathcal{B}(w,\mathscr{A})\nabla_\Gamma \eta \cdot \nabla_\Gamma \chi \,\mathrm{d}\sigma,\end{aligned} \tag{4.11}$$

where $\mathcal{B}(w,\mathscr{A})$ is given by

$$\mathcal{B}(w,\mathscr{A}) = \partial^\bullet \mathscr{A} + \nabla_\Gamma \cdot w \mathscr{A} - 2D(w) \tag{4.12}$$

and $D(w)$ is the rate of deformation tensor

$$D(w)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1} \mathscr{A}_{ik}\underline{D}_k w_j + \mathscr{A}_{jk}\underline{D}_k w_i \qquad \text{for } i,j = 1,\ldots,n+1.$$

Let $\mathscr{B} = \mathscr{B}(x,t)$ be a smooth vector field which is tangent to $\Gamma(t)$. Then for $\eta \in C^1_{\mathcal{H}}$, $\eta \in C^1_{\mathcal{V}}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathscr{B}\eta \cdot \nabla_\Gamma \chi \,\mathrm{d}\sigma &= \int_{\Gamma(t)} \mathscr{B}\partial^\bullet \eta \cdot \nabla_\Gamma \chi + \mathscr{B}\eta \cdot \nabla_\Gamma \partial^\bullet \chi \,\mathrm{d}\sigma \\ &\quad + \int_{\Gamma(t)} \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B})\eta \cdot \nabla_\Gamma \chi \,\mathrm{d}\sigma,\end{aligned} \tag{4.13}$$

where $\mathcal{B}_{\mathrm{adv}}(w,\mathscr{B})$ is given by

$$\mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}) = \partial^\bullet \mathscr{B} + \mathscr{B}\nabla_\Gamma \cdot w - \sum_{j=1}^{n+1} \mathscr{B}_j \underline{D}_j w.$$

The identity (4.13) is equivalent to Lemma A.1 in (Elliott and Venkataraman, 2015). The proof of (4.11) and (4.13) follows from applying (4.10) with the identity

$$\partial^\bullet \underline{D}_i \chi = \underline{D}_i \partial^\bullet \chi - \sum_{j=1}^{n+1} \underline{D}_i w_j \underline{D}_j \chi + \left(\nabla_\Gamma (w \cdot \nu) \cdot \nabla_\Gamma \chi - \sum_{j,l=1}^{n+1} w_j \underline{D}_l \chi \underline{D}_l \nu_j \right) \nu_i. \tag{4.14}$$

A useful intermediate result in the calculation of (4.14) is the identity:

$$\partial^\bullet \nu_i(x,t) = \partial^\bullet \partial_i d(x,t) = -\nabla_\Gamma (w(x,t) \cdot \nu(x,t)) + \mathcal{H}(x,t) w(x,t), \tag{4.15}$$

for $x \in \Gamma(t), i = 1,\ldots,n+1$.

4.2.2. *Evolving flat domain.* In the case of a flat domain with boundary that is a bounded open set $\Omega(t)$ of $\mathbb{R}^{n+1}$, we impose that the flow is defined on the closure of $\Omega(t)$, so that $\Phi_t \colon \bar\Omega_0 \to \bar\Omega(t)$ and $\Phi_t(\Omega_0) = \Omega(t)$ and $\Phi_t(\Gamma_0) = \Gamma(t)$. The map Φ_t defines a velocity field w on $\bar\Omega_T$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x,t) = w(\Phi_t(x,t),t) \qquad \text{for } x \in \bar\Omega_0.$$

We assume that

$$\begin{aligned}&\Phi_t \colon \Omega_0 \to \Omega(t) \text{ is a } C^2\text{-diffeomorphism} \\ &\Phi_t \colon \Gamma_0 \to \Gamma(t) \text{ is a } C^2\text{-diffeomorphism} \\ &\Phi_t \in C^2([0,T] \times \bar\Omega(t)) \\ &\sup_{\Omega(t)} |\nabla \cdot w| + \sup_{\Gamma(t)} |\nabla_\Gamma \cdot w| < C \quad \text{for all } t \in [0,T].\end{aligned}$$

Remark 4.5. We may think of $\{\Omega(t)\}_{t\in[0,T]}$ as an evolving subset with boundary of the flat hypersurface $\left\{x \in \mathbb{R}^{n+2} : x_{n+2} = 0\right\}$. with boundary. We set

$$\Upsilon(t) := \left\{(x,0) \in \mathbb{R}^{n+2} : x \in \Omega(t)\right\} \qquad \text{for } t \in [0,T].$$

Then $\{\Upsilon(t)\}_{t\in[0,T]}$ is a smooth hypersurface with boundary.

Let $\mathcal{V}(t) = H^1(\Omega(t))$ and $\mathcal{H}(t) = L^2(\Omega(t))$. We denote by $\mathcal{V}^*(t) = (H^1(\Omega(t)))^*$. For each $t \in [0,T]$, $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))$ form a separable Hilbert triple. We define the push-forward operator ϕ_t by

$$(\phi_t \eta)(x,t) := \eta(\Phi_{-t}(x)) \qquad \text{for } \eta \in \mathcal{H}_0, x \in \Omega(t). \tag{4.16}$$

Our assumptions imply that $(L^2(\Omega(t)), \phi_t)_{t\in[0,T]}$ and $(H^1(\Omega(t)), \phi_t)_{t\in[0,T]}$ are both compatible pairs and the spaces $L^2_{\mathcal{H}}$ and $L^2_{\mathcal{V}}$ are well defined (Alphonse et al., 2015b, Section 4.2).

The push-forward operator allows us to define a material derivative from (2.1). For $\eta \in C^1_{\mathcal{H}}$, we define the strong material derivative $\partial^\bullet \eta$ which can be characterised by

$$\partial^\bullet \eta = \partial_t \widetilde{\eta} + w \cdot \nabla \widetilde{\eta},$$

where $\widetilde{\eta}$ is an extension of η to $\Omega_T \cup \mathcal{N}_T$. We can show that Assumption 2.4 holds so that the transport formula for the $\mathcal{H}(t)$-inner product holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \eta \varphi \,\mathrm{d}x = \int_{\Omega(t)} \partial^\bullet \eta \varphi + \eta \partial^\bullet \varphi \,\mathrm{d}x + g(t;\eta,\varphi) \qquad \text{for } \eta, \varphi \in C^1_{\mathcal{H}},$$

where

$$g(t;\eta,\varphi) = \int_{\Omega(t)} \eta \varphi \nabla \cdot w \,\mathrm{d}x.$$

Furthermore, we have transport formula for a Dirichlet inner product and advection inner product. Let $\mathscr{A}(\cdot,t)\colon \Omega(t) \to \mathbb{R}^{(n+1)\times(n+1)}$ be a smooth, symmetric, uniformly positive definite diffusion tensor then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \mathscr{A} \nabla \eta \cdot \nabla \varphi \,\mathrm{d}x = {} & \int_{\Omega(t)} \mathscr{A} \nabla \partial^\bullet \eta \cdot \nabla \varphi + \mathscr{A} \nabla \eta \cdot \nabla \partial^\bullet \varphi \,\mathrm{d}x \\ & + \int_{\Omega(t)} \mathcal{B}(w,\mathscr{A}) \nabla \eta \cdot \nabla \varphi \,\mathrm{d}x \qquad \text{for all } \eta, \varphi \in C^1_{\mathcal{V}},\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}(w,\mathscr{A}) &= \partial^\bullet \mathscr{A} + \mathscr{A} \nabla \cdot w - 2D(w) \\ D(w)_{ij} &= \frac{1}{2} \sum_{k=1}^{n+1} \mathscr{A}_{ik} \partial_{x_k} w_j + \mathscr{A}_{jk} \partial_{x_k} w_i \qquad \text{for } i,j = 1,\ldots,n+1.\end{aligned}$$

Let $\mathscr{B}(\cdot,t)\colon \Omega(t) \to \mathbb{R}^{n+1}$ be a smooth vector field then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \mathscr{B} \eta \cdot \nabla \varphi \,\mathrm{d}x = {} & \int_{\Omega(t)} \mathscr{B} \partial^\bullet \eta \cdot \nabla \varphi + \mathscr{B} \nabla \eta \cdot \nabla \partial^\bullet \varphi \,\mathrm{d}x \\ & + \int_{\Omega(t)} \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}) \eta \cdot \nabla \varphi \,\mathrm{d}x \qquad \text{for all } \eta \in C^1_{\mathcal{H}}, \varphi \in C^1_{\mathcal{V}},\end{aligned}$$

where

$$\mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}) = \partial^\bullet \mathscr{B} + \mathscr{B} \nabla \cdot w - \sum_{j=1}^{n+1} \mathscr{B}_j \partial_{x_j} w.$$

4.3. **Tangential and material velocities.** For a hypersurface defined by a level set function $\Gamma(t) := \{x \in \mathbb{R}^{n+1} : \Psi(x,t) = 0\}$, we can define a *normal velocity* by

$$w_\nu := -\frac{\Psi_t}{|\nabla \Psi|}\nu.$$

This is sufficient to define the evolution of the hypersurface from Γ_0. If the (possibly flat) hypersurface has a boundary, we also require a conormal velocity of the boundary.

On the other hand in applications there maybe a physical *material velocity*

$$w = w_\nu + w_\tau,$$

where w_ν is the normal component and w_τ is the tangential component. The tangential velocity is associated with the transport of material points on the surface of $\Gamma(t)$. Often, a material velocity is defined through a parametrisation of the domain.

When evolving a computational domain $\{\Gamma_h(t)\}$ for a finite element method, we should ensure that the quality of the mesh is preserved. Poor mesh quality leads to large errors and poorly conditioned system of linear equations to solve. One possibility is to add an extra *arbitrary tangential velocity* that moves nodes to ensure mesh quality changing the parameterisation of the surface. This can lead to a different formulation of the partial differential equation we wish to solve (Elliott and Styles, 2012).

Remark 4.6. In this work, we are not concerned with the issue of generating moving meshes in order to define a good subdivision and assume that such a velocity is given. Previous studies, such as the work of Elliott and Fritz (2016), which have studied this problem, can be included into our framework.

Let $\{\Gamma(t)\}$ be a closed evolving surface defined by a parameterisation $G_{\mathrm{phys}}(\cdot,t) \colon \Gamma_0 \to \mathbb{R}^{n+1}$ so that $\Gamma(t) = G_{\mathrm{phys}}(\Gamma_0,t)$ where the velocity by $w_{\mathrm{phys}}(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} G_{\mathrm{phys}}(x_0,t)|_{G_{\mathrm{phys}}(x_0)=x}$ has a tangential component which is physical in the sense that it transports matter.. For $t \in [0,T]$, let $\mathcal{M}(t) \subset \Gamma(t)$ be the evolution of a portion $\mathcal{M}_0 \subset \Gamma_0$ under the parametrisation, i.e. $\mathcal{M}(t) = G_{\mathrm{phys}}(\mathcal{M}_0,t)$. If G_{phys} is sufficiently smooth, we can define a linear homeomorphism $\phi_t^{\mathrm{phys}} \colon L^2(\Gamma_0) \to L^2(\Gamma(t)$ and be able to define both a material derivative, for which we use the notation $\partial^\bullet_{\mathrm{phys}}$, and use transport formula.

Consider a (time-dependent) scalar concentration field u on $\Gamma(t)$ which satisfies a conservation law of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{M}(t)} u \,\mathrm{d}\sigma = \int_{\mathcal{M}(t)} s \,\mathrm{d}\sigma - \int_{\partial \mathcal{M}(t)} q \cdot \mu \,\mathrm{d}\sigma,$$

where s represents a source/sink of u per unit volume in $\mathcal{M}(t)$ and q the flux of u across the boundary of $\mathcal{M}(t)$. Note that the advective transport of matter across $\partial\mathcal{M}(t)$ is accounted for by the tangential velocity of the parameterisation. We assume that q is a tangential only diffusive flux ($q \cdot \nu = 0$). Using the transport formula (4.10), the left hand side is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{M}(t)} u \,\mathrm{d}\sigma = \int_{\mathcal{M}(t)} \partial^\bullet_{\mathrm{phys}} u + u \nabla_\Gamma \cdot w_{\mathrm{phys}} \,\mathrm{d}\sigma,$$

and applying the integration by parts formula (4.5) to the boundary term, we see

$$-\int_{\partial \mathcal{M}(t)} q \cdot \mu \,\mathrm{d}\sigma = -\int_{\mathcal{M}(t)} \nabla_\Gamma \cdot q \,\mathrm{d}\sigma.$$

Hence, we have derived a pointwise conservation law for u:

$$\partial^\bullet_{\mathrm{phys}} u + u \nabla_\Gamma \cdot w_{\mathrm{phys}} = s - \nabla_\Gamma \cdot q. \tag{4.17}$$

A typical choice of q is $q = -\nabla_\Gamma u$ and $s = 0$. This leads to the advection diffusion equation on an evolving surface:

$$\partial^\bullet_{\mathrm{phys}} u + u \nabla_\Gamma \cdot w_{\mathrm{phys}} = \Delta_\Gamma u.$$

Next, we suppose we have a different parametrisation $G(\cdot,t) \colon \Gamma_0 \to \mathbb{R}^{n+1}$ so that $\Gamma(t) = G(\Gamma_0,t)$. This new parametrisation induces a new velocity field w given by $w(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} G(x_0,t)|_{G(x_0)=x}$. We can decompose $w = w_{\mathrm{phys}} + w_A$ where w_A is an extra arbitrary tangential velocity, $w_A \cdot \nu = 0$. Since we only add a tangential component, the normal components of each velocity agree $w_{\mathrm{phys}} \cdot \nu = w \cdot \nu$. Given sufficient smoothness assumptions, the parametrisation G defines a linear homeomorphism $\phi_t \colon L^2(\Gamma_0) \to L^2(\Gamma(t))$ and therefore material derivatives which we will denote by $\partial^\bullet$ which satisfies

$$\partial^\bullet u = \widetilde{u}_t + \nabla \widetilde{u} \cdot w = \widetilde{u}_t + \nabla \widetilde{u} \cdot (w_{\mathrm{phys}} + w_A) = \partial^\bullet_{\mathrm{phys}} u + \nabla_\Gamma u \cdot w_A.$$

We also see that

$$\begin{aligned}
\int_{\Gamma(t)} \partial^\bullet_{\mathrm{phys}} u\varphi + u\varphi \nabla_\Gamma \cdot w_{\mathrm{phys}} \,\mathrm{d}\sigma &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi \,\mathrm{d}\sigma - \int_{\Gamma(t)} u \partial^\bullet_{\mathrm{phys}} \varphi \,\mathrm{d}\sigma \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi \,\mathrm{d}\sigma - \int_{\Gamma(t)} u \partial^\bullet \varphi - u \nabla_\Gamma \varphi \cdot w_A \,\mathrm{d}\sigma \\
&= \int_{\Gamma(t)} \partial^\bullet u \varphi + u\varphi \nabla_\Gamma \cdot w \,\mathrm{d}\sigma + \int_{\Gamma(t)} u \nabla_\Gamma \varphi \cdot w_A \,\mathrm{d}\sigma.
\end{aligned}$$

This implies that we can transform a partial differential equation given by the conservation law (4.17) into a weak form given by

$$\int_{\Gamma(t)} \partial^\bullet u\varphi + u\varphi \nabla_\Gamma \cdot w - q \cdot \nabla_\Gamma \varphi - s\varphi \,\mathrm{d}\sigma + \int_{\Gamma(t)} w_A u \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma = 0,$$

or variational form given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi \,\mathrm{d}\sigma - \int_{\Gamma(t)} q \cdot \nabla_\Gamma \varphi + s\varphi \,\mathrm{d}\sigma + \int_{\Gamma(t)} w_A u \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma = \int_{\Gamma(t)} u \partial^\bullet \varphi \,\mathrm{d}\sigma.$$

By defining q_A by

$$q_A = q - u w_A,$$

we recover a the same conservation law structure. For example, if $q = -\nabla_\Gamma u$, then the evolving surface advection diffusion equation becomes:

$$\partial^\bullet u + u \nabla_\Gamma \cdot w = \Delta_\Gamma u + \nabla_\Gamma \cdot (w_A u).$$

In this work, we will consider a general setting in which the transformed equation is

$$\partial^\bullet u + u \nabla_\Gamma \cdot w = \nabla_\Gamma \cdot (\mathscr{A} \nabla_\Gamma u) + \nabla_\Gamma \cdot (\mathscr{B} u) + \mathscr{C} u.$$

5. Evolving surface finite element spaces

In this section, we will give precise definitions of the evolving surface finite element space we use. The key idea is to define a single surface finite element including the cases of affine finite elements and isoparametric finite elements and then proceed to show the assumptions required in order to make sense of these structures in an evolving context. We will also define abstract lift operators which will relate a finite element space $\mathcal{V}_h(t)$ to the smooth problem spaces $\mathcal{V}(t)$.

The authoritative text of Ciarlet (1978) gives guidance on the minimum requirements of a finite element method. The first stage is to define admissible partition of the computational domain. In our case the computational domain will be an approximation of the exact domain in which the equations are posed. The second basic aspect is that we should use element-wise defined function spaces which contain polynomials or are close to polynomials in a certain sense. This allows use of standard interpolation theorems to ensure accuracy and simple quadrature rules to perform integrals. Finally, there should be at least one canonical basis of the global finite element functions which have small support and simple definitions. Our extensions build on the work of Nedelec (1976), Dziuk (1988) and Heine (2005) for surfaces and earlier work by Ciarlet and Raviart (1972) and Bernardi (1989) for Cartesian domains.

Throughout this section we will denote global discrete quantities with a subscript $h \in (0,h_0)$, which is related to element size. We assume implicitly that these structures exist for each h in this range.

5.1. **Surface finite elements.** We first consider a single stationary surface finite element. Roughly speaking, we think of surface finite elements as an element parametrised over a reference finite element.

Definition 5.1 (Reference finite element). The triple (K,P,Σ) is a *reference finite element* if:

- $K \subset \mathbb{R}^n$ is a closed domain with Lipschitz piecewise smooth boundary. We call K the *element domain*.
- P is a finite dimensional space of functions over K. We call P the *shape functions*.
- $\Sigma = \{\sigma_1,\dots,\sigma_d\}$ is a basis of P' the dual space to P. We call Σ the *nodal variables* or *degrees of freedom*.

We say that Σ determines P if for $\chi \in P$ with $\sigma(\chi) = 0$ for all $\sigma \in \Sigma$, we have $\chi = 0$. We will often write (K,P^K,Σ^K) for a finite element with element domain K.

As part of this definition, we are implicitly assuming that the nodal variables live in the dual to a larger function space than P. We will see that this usually requires further smoothness or continuity of finite element functions. We give an example of a simplical finite element, but this definition includes other examples such as iso-parametric finite elements and brick finite elements.

Example 5.2. In $\mathbb{R}^n$, a (non-degenerate) *n-simplex* is the convex hull K of $n+1$ points $\{a_i\}_{i=1}^{n+1} \subset \mathbb{R}^n$, called the *vertices* of the n-simplex, which are not contained in a common $(n-1)$-dimension hyperplane. More precisely, we have

$$K = \left\{ x = \sum_{i=1}^{n+1} \lambda_i a_0 : 0 \le \lambda_i \le 1, 1 \le i \le n+1, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

For each $x \in K$, we call $\{\lambda_i\}_{i=1}^{n+1}$ *barycentric coordinates*.

For any integer l with $0 \le l \le n$, an l-facet of an n-simplex K is any l-simplex whose $(l+1)$ vertices are also vertices of K. We call a $(n-1)$-facet a *boundary facet*. We will also use the term boundary facet for any boundary polytopes (union of simplicies) of a polytope K.

For each $k \ge 0$, we shall denote by P_k the space of all polynomials of degree k in the variables $x_1,\dots,x_n$ in $\mathbb{R}^n$. For any set $A \subset \mathbb{R}^n$, we let

$$P_k(A) = \{\chi|_A : \chi \in P_k\}.$$

The space $P_1(K)$ consists of affine functions over a simplex K. We can construct a standard piecewise linear finite element $(K,P_1(K),\Sigma^K)$ by choosing $\Sigma^K = \{\chi \mapsto \chi(a_i) : 1 \le i \le n+1\}$. We can also define higher order spaces $(K,P_k(K),\Sigma^K)$, for $k \ge 2$, by including extra evaluation points in Σ^K (see, for example, Ciarlet (1978, Section 2.2)).

We next define a surface finite element which takes its inspiration from the notion of curved finite elements studied by Ciarlet and Raviart (1972) and Bernardi (1989). The key different here is that a surface finite element is an n-dimensional parameterised surface embedded in $\mathbb{R}^{n+1}$ with boundary.

Definition 5.3 (Surface finite element). Let $(\hat{K},\hat{P},\hat{\Sigma})$ be a reference finite element with $\hat{K}\subset\mathbb{R}^n$. Let $F_K\colon \hat{K}\to\mathbb{R}^{n+1}$ satisfy

(1) (a) $F_K\in C^1(\hat{K},\mathbb{R}^{n+1})$;
 (b) $\operatorname{rank}\nabla F_K = n$;
 (c) F_K is a bijection onto its image;

(2) F_K can be decomposed into an affine part and smooth part

$$F_K(\hat{x}) = A_K\hat{x} + b_K + \Phi_K(\hat{x})$$

such that $\Phi_K\in C^1(\hat{K})$

$$C_K := \sup_{\hat{x}\in\hat{K}}\left\|\nabla\Phi_K(\hat{x})A_K^\dagger\right\| < 1, \tag{5.1}$$

where $A^\dagger$ denotes the pseudo-inverse of A.

Let (K,P,Σ) be the triple given by

$$\begin{aligned} K &:= F_K(\hat{K})\\ P &:= \{\hat{\chi}\circ F_K^{-1} : \hat{\chi}\in\hat{P}\}\\ \Sigma &:= \{\chi\mapsto\hat{\sigma}(\chi\circ F_K) : \hat{\sigma}\in\hat{\Sigma}\}. \end{aligned}$$

Under the above assumptions, we call (K,P,Σ) a *surface finite element* and call $(\hat{K},\hat{P},\hat{\Sigma})$ the associated *reference finite element*.

For any matrix A of full column rank, the pseudoinverse of A is given by

$$A^\dagger = (A^tA)^{-1}A^t,$$

and this particular psuedoinverse constitutes a left inverse $A^\dagger A = \mathrm{Id}$. We note that our assumptions implies that both A_K and $\nabla F_K(\cdot)$ are both of full column rank.

The first three assumptions in the definition of surface finite element imply that K is a parametrised surface and the fourth (5.1) that K is not too curved. The final assumption allows the case that $\hat{K}$ is a flat simplical domain and K is curved.

Remark 5.4. We denote by ν_K the unit normal vector field to K is the unique (up to sign) unit vector orthogonal to the $\hat{x}_i$ partial derivatives of F_K for $i=1,\dots,n$ given by

$$\nu_K := \begin{cases} \dfrac{\left(\frac{\partial F_K}{\partial\hat{x}_1}\right)^\perp}{\left|\left(\frac{\partial F_K}{\partial\hat{x}_1}\right)^\perp\right|} & \text{for } n=1\\[2ex] \dfrac{\left(\frac{\partial F_K}{\partial\hat{x}_1}\wedge\ldots\wedge\frac{\partial F_K}{\partial\hat{x}_n}\right)}{\left|\left(\frac{\partial F_K}{\partial\hat{x}_1}\wedge\ldots\wedge\frac{\partial F_K}{\partial\hat{x}_n}\right)\right|} & \text{for } n>1. \end{cases}$$

The sign of normal vector field is chosen by fixing a permutation of the barycentric coordinates $\hat{x}_1,\dots\hat{x}_n$ of the reference element. By swapping any two elements, we reverse the sign of ν_K. For a simplex reference element, the orientation can be fixed by ordering the labels of vertices so that $a_i = F_K(\hat{a}_i)$ where $\{\hat{a}_i\}$ are the vertices of the reference element domain.

Definition 5.5. Let $\theta\in\mathbb{N}$. We say that (K,P,Σ) is a θ-surface finite element if

(1) $F_K \in C^\theta(\hat{K})$ (i.e. K is a C^θ-hypersurface);
(2) the space P contains the functions $\hat{\chi} \circ F_K^{-1}$ for all $\hat{\chi} \in P_\theta(\hat{K})$;
(3) the space P is contained in $C^{\theta+1}(K)$.

For a θ-surface finite element (K,P,Σ), we have that the Sobolev space $W^{k,p}(K)$ is well defined for $0 \le k \le \theta$ and $1 \le p \le \infty$.

Remark 5.6. For a surface finite element (K,P,Σ), the first fundamental form is given by

$$G(\hat{x}) = (g_{ij}(\hat{x})), \qquad g_{ij}(\hat{x}) = \partial_{\hat{x}_i} F_K(\hat{x}) \partial_{\hat{x}_j} F_K(\hat{x}).$$

Using transformation formulae (4.6) and (4.7), we have for $\chi \in P$

$$\int_K \chi \,\mathrm{d}\sigma = \int_{\hat{K}} \hat{\chi}(\hat{x}) \sqrt{g(\hat{x})}\,\mathrm{d}\hat{x}, \qquad \nabla_K \chi(x) = \nabla F_K(\hat{x})^t G^{-1}(\hat{x}) \nabla \hat{\chi}(\hat{x}),$$

where $F_K(\hat{x}) = x$ and $\hat{\chi}(\hat{x}) = \chi(x)$, and

$$G(\hat{x}) = \big(\nabla F_K(\hat{x})\big)^t \big(\nabla F_K(\hat{x})\big), \qquad g(\hat{x}) = \det G(\hat{x}).$$

Example 5.7. We are thinking of three particular examples. The first is due to Dziuk (1988) and the second due to Heine (2005).

(1) Let $(\hat{K}, P_1(\hat{K}), \hat{\Sigma})$ be a reference Lagrangian finite element. Consider the affine map $F_K \colon \hat{K} \to \mathbb{R}^{n+1}$ given by $F_K(\hat{x}) = A_K \hat{x} + b_K$. If A_K is non-degenerate, then this defines a surface finite element (K,P,Σ). The element domain K is determined by its vertices and P consists of affine functions over K. This is the surface finite element introduced by Dziuk (1988) and we will call this an *affine finite element*. We think of line segments embedded in $\mathbb{R}^2$, triangles embedded in $\mathbb{R}^3$, and tetrahedra embedded in $\mathbb{R}^4$.
(2) Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a reference finite element. Let (K,P,Σ) be a surface finite element which the image of $(\hat{K}, \hat{P}, \hat{\Sigma})$ under a map F_K which satisfies $F_K \in (\hat{P})^{n+1}$. We call (K,P,Σ) an isoparametric (surface) finite element. This construction is a generalisation of an affine finite element and was introduced by Heine (2005). We note that the functions in P will not necessarily consist of polynomials over K even if $\hat{P}$ consists of polynomials over $\hat{K}$, however this leads to a practical scheme where integrals are computed over reference elements.
(3) Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a reference finite element. Then $(\hat{K}, \hat{P}, \hat{\Sigma})$ can be thought of a surface finite element (K,P,Σ) by defining the parametrisation F_K by

$$F_K(\hat{x}) = (\hat{x}_1, \ldots, \hat{x}_n, 0).$$

Note that $\hat{K} \subset \mathbb{R}^n$ but $K \subset \mathbb{R}^{n+1}$. In general, we will consider flat surface finite elements to be surface finite elements to have parametrisation F_K such that $(F_K)_{n+1} \equiv 0$. In this case, we will consider the identification that $K \subset \mathbb{R}^n$ and F_K is the first n components of the full F_K.

Examples of each of these first two cases is shown in Figure 1.

Lemma 5.8. *Let (K,P,Σ) be a surface finite element parameterised by $F_K \colon \hat{K} \to K$. The mapping F_K is a C^1-diffeomorphism and satisfies*

$$\sup_{\hat{x} \in \hat{K}} \|\nabla F_K(\hat{x})\| \le (1 + C_K) \|A_K\| \tag{5.2}$$

$$\sup_{\hat{x} \in \hat{K}} \left\| (\nabla F_K)^\dagger(\hat{x}) \right\| \le (1 - C_K) \left\| A_K^\dagger \right\|, \tag{5.3}$$

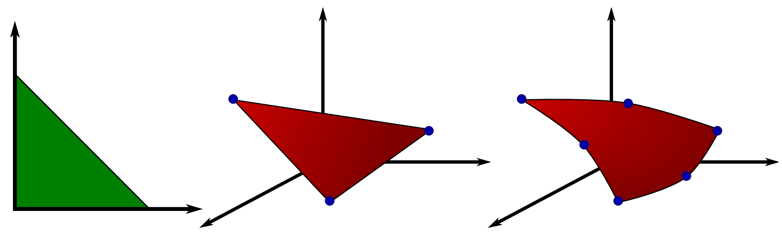

FIGURE 1. Examples of different finite elements in the case $n = 2$. Left shows a reference finite element (in green), center shows an affine finite element and right shows an isoparametric finite element with a quadratic F_K. The plot shows the element domains in red and the location of nodes in blue.

and also for all $\hat{x} \in \hat{K}$

$$(1 - C_K)^{2n} \det(A_K^t A_K) \leq g(\hat{x}) \leq (1 + C_K)^{2n} \det(A_K^t A_K). \tag{5.4}$$

Proof. The proof of (5.2), (5.3) and (5.4) follows immediately from (5.1) by writing ∇F_K as

$$\nabla F_K(\hat{x}) = (\mathrm{Id} + \nabla\Phi_K(\hat{x}) A_K^{\dagger}) A_K. \qquad \square$$

Lemma 5.9. *Let (K, P, Σ) be a surface finite element parameterised over a reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$ by F_K. Denote by*

$$h_K = \mathrm{diam}(K)$$
$$\rho_K = \sup\{\mathrm{diam}(B) : B \textit{ is a n-dimensional ball contained in } K\}.$$

To simplify notation, we will write $\hat{h}$ and $\hat{\rho}$ for $h_{\hat{K}}$ and $\rho_{\hat{K}}$. Then we have that

$$\|A_K\| \leq (1 - C_K)^{-1} \frac{h_K}{\hat{\rho}} \tag{5.5a}$$

$$\left\|A_K^{\dagger}\right\| \leq \left(1 - \frac{C_K}{1 - C_K} \frac{\hat{h}}{\hat{\rho}}\right)^{-1} \frac{\hat{h}}{\rho_K} \tag{5.5b}$$

$$\sup_{\hat{x} \in \hat{K}} \sqrt{g(\hat{x})} \leq \frac{1}{\mathrm{meas}\,\hat{K}} \left(\frac{1 + C_K}{1 - C_K}\right)^n \mathrm{meas}(K). \tag{5.5c}$$

Remark 5.10. We note that the volume of an element $\mathrm{meas}(K)$ can be estimated by h_K and ρ_K by

$$c_1 \rho_K^n \leq \mathrm{meas}\, K \leq c_2 h_K^n.$$

Here the positive constants c_1, c_2 depend on the volume of the unit ball in $\mathbb{R}^n$ and the constant C_K.

Proof. To show (5.5a), we start by noticing that

$$\|A_K\| = \frac{1}{\hat{\rho}} \sup\{|A_K \xi| : \xi \in \mathbb{R}^n, |\xi| = \hat{\rho}\}.$$

From the definition of $\hat{\rho}$ we know that for all $\xi \in \mathbb{R}^n$, $|\xi| = \hat{\rho}$, there exists $\hat{y}, \hat{z} \in \hat{K}$ such that $\hat{y} - \hat{z} = \xi$. Then we have from the expansion of F_K that

$$|A_K \xi| = |A_K(\hat{y} - \hat{z})| \leq |F_K(\hat{y}) - F_K(\hat{z})| + |\Phi_K(\hat{y}) - \Phi_K(\hat{z})|.$$

Since $\hat{y}, \hat{z} \in \hat{K}$, $F_K(\hat{y}), F_K(\hat{z}) \in K$, thus we infer

$$|F_K(\hat{y}) - F_K(\hat{z})| \leq h_K.$$

Using the smoothness of Φ_K and the bound (5.1) we have

$$|\Phi_K(\hat{y}) - \Phi_K(\hat{z})| \leq \sup_{\hat{x}\in\hat{K}} \|D\Phi_K\| \, \hat{\rho} \leq C_K \|A_K\| \, \hat{\rho}.$$

Combining the above estimates results in (5.5a).

For (5.5b) we proceed in a similar fashion with

$$\left\| A_K^\dagger \right\| = \frac{1}{\rho_K} \sup \left\{ \left| A_K^\dagger \xi \right| : \xi \in \mathbb{R}^{n+1}, |\xi| = \rho_K \right\}.$$

For all $\xi \in \mathbb{R}^{n+1}$ with $|\xi| = \rho_K$, there exists $y, z \in K$ such that $A_K^\dagger \xi = A_K^\dagger (y - z)$ and $|y - z| = \rho_K$. Since F_K is bijective, there exists $\hat{y}, \hat{z} \in \hat{K}$ such that $F_K(\hat{y}) = y$ and $F_K(\hat{z}) = z$. Then we have using the decomposition of F_K that

$$A_K^\dagger \xi = A_K^\dagger (y - z) = A_K^\dagger (F_K(\hat{y}) - F_K(\hat{z})) = A_K^\dagger A_K (\hat{y} - \hat{z}) + A_K^\dagger (\Phi_K(\hat{y}) - \Phi_K(\hat{z})).$$

Using the fact that $A_K^\dagger A_K$ is the identity map and a similar calculation as above for the Φ_K term we have

$$\left\| A_K^\dagger \right\| \leq \frac{1}{\rho_K} \left(\hat{h} + \frac{C_K}{1 - C_K} \frac{h_K}{\hat{\rho}} \hat{h} \left\| A_K^\dagger \right\| \right).$$

Rearranging gives (5.5b).

To see (5.5c) we apply each of the previous two bounds with the result of Lemma 5.8 to see

$$\begin{aligned}
\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} &\leq (1 + C_K)^n \sqrt{\det(A_K^t A_K)} \\
&\leq \frac{1}{\operatorname{meas} \hat{K}} (1 + C_K)^n \int_{\hat{K}} \sqrt{\det(A_K^t A_K)} \, \mathrm{d}\hat{x} \\
&\leq \frac{1}{\operatorname{meas} \hat{K}} (1 + C_K)^n / (1 - C_K)^n \int_{\hat{K}} \sqrt{g(\hat{x})} \, \mathrm{d}\hat{x} \\
&\leq \frac{1}{\operatorname{meas} \hat{K}} \left(\frac{1 + C_K}{1 - C_K} \right)^n \operatorname{meas}(K) \\
&\leq C(\hat{K}) \operatorname{meas}(K).
\end{aligned}$$

□

This scaling property allows us to characterise Sobolev spaces over a surface finite element K and calculate norms over $\hat{K}$. We note that if K is a θ-surface finite element, we may define surface Sobolev spaces over K with up to θ weak derivatives.

Lemma 5.11. *Let (K, P, Σ) be a θ-surface finite element parameterised by F_K over $\hat{K}$. Let $0 \leq m \leq \theta$ and $p \in [1, \infty]$, then $\chi \in W^{m,p}(K)$ implies $\hat{\chi} = \chi \circ F_K$ belongs to $W^{m,p}(\hat{K})$. We have for any $\chi \in W^{m,p}(K)$ that*

$$|\hat{\chi}|_{W^{m,p}(\hat{K})} \leq c \left(\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} \right)^{-\frac{1}{p}} \|A_K\|^m \, |\chi|_{W^{m,p}(K)}. \tag{5.6}$$

We also have for any $\hat{\chi} \in W^{1,p}(\hat{K})$ that $\chi = \hat{\chi} \circ F_K^{-1} \in W^{1,p}(K)$ and

$$|\chi|_{W^{m,p}(K)} \leq c \left(\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} \right)^{\frac{1}{p}} \left\| A_K^\dagger \right\|^m |\hat{\chi}|_{W^{m,p}(\hat{K})}. \tag{5.7}$$

Proof. First, assume that $\chi \in C^m(\bar{K})$ so that $\hat{\chi} \in C^m(\bar{\hat{K}})$. Let $\alpha \in \mathbb{N}_0^n$ be a multi-index then we have for almost all $x = F_K(\hat{x}) \in K$

$$D^\alpha \hat{\chi}(\hat{x}) = D^m \hat{\chi}(\hat{x})\big(\hat{e}_1, \dots, \hat{e}_m\big) = D_K^m \chi(x)\big(DF_K(\hat{x})\hat{e}_1, \dots, DF_K(\hat{x})\hat{e}_m\big),$$

where $\hat{e}_l$, $l = 1, \dots, n$ are the coordinate directions in $\mathbb{R}^n$. This identity follows by a simple induction argument using the definition of tangential gradient (4.6)

$$(\nabla_K \chi(x)) \cdot (DF_K(\hat{x})\hat{e}_l) = \sum_{i,j=1}^n g^{ij}(\hat{x}) \frac{\partial F_K(\hat{x})}{\partial \hat{x}_i} \frac{\partial \hat{\chi}(\hat{x})}{\partial \hat{x}_j} \cdot \frac{\partial F_K(\hat{x})}{\partial \hat{x}_l} = \nabla \hat{\chi}(\hat{x}) \cdot \hat{e}_l.$$

Then applying the integral transformation rule (4.7) gives (5.6). We transform from $C^m(\bar{K})$ to $W^{k,p}(K)$ using a density argument to get

$$|\hat{\chi}|_{W^{m,p}(\hat{K})} \leq c \left(\sup_{\hat{x} \in \hat{K}} \sqrt{g(\hat{x})} \right)^{-\frac{1}{p}} \sup_{\hat{x} \in \hat{K}} \|\nabla F_K(\hat{x})\|^m \, |\chi|_{W^{m,p}(K)} .$$

The final result follows from Lemma 5.8.

The converse results (5.7) follow in a similar fashion using the identity that for a coordinate direction e_l in $\mathbb{R}^{n+1}$, using the same notation as above, we have

$$\nabla_K \chi(x) \cdot e_l = \nabla \hat{\chi}(\hat{x}) \cdot \big(\nabla F_K(\hat{x})\big)^\dagger e_l. \qquad \square$$

Given a surface finite element (K, P, Σ), let $\{\chi_i : 1 \leq i \leq d\} \subset P$ be the basis dual to Σ. This is the set of *basis functions* of the finite element. If η is a function for which all $\sigma_i(\eta)$, $1 \leq i \leq d$ is well defined, then we define the *local interpolant* by

$$I_K \eta := \sum_{i=1}^m \sigma_i(\eta) \chi_i. \tag{5.8}$$

We can think of $I_K \eta$ as the unique shape function that has the same nodal values as η so that, in particular, $I_K \chi = \chi$ for $\chi \in P$.

Theorem 5.12 (Local interpolation estimate)**.** *Let (K, P, Σ) be a θ-surface finite element parameterised over a reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$ by F_K. Let the following inclusions hold for $0 \leq m \leq \theta$, $1 \leq k+1 \leq \theta$ and $p, q \in [1, \infty]$,*

$$W^{k+1,p}(\hat{K}) \hookrightarrow C(\hat{K})$$
$$W^{k+1,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K})$$
$$P_k(\hat{K}) \subset \hat{P} \subset W^{m,q}(\hat{K}).$$

Then there exists a constant $C = C(\hat{K}, \hat{P}, \hat{\Sigma})$ such that for all functions $\chi \in W^{k+1,p}(K)$

$$|\chi - I_K \chi|_{W^{m,q}(K)} \leq C \operatorname{meas}(K)^{1/q - 1/p} \frac{h_K^{k+1}}{\rho_K^m} \, |\chi|_{W^{k+1,p}(K)} . \tag{5.9}$$

Proof. Under the above assumptions on the reference finite element we have a Bramble-Hilbert Lemma (Ciarlet, 1978, Theorem 3.1.4) that there exists a constant $C = C(\hat{K}, \hat{P}, \hat{\Sigma}) > 0$ such that for all functions $\hat{\chi} \in W^{k+1,p}(\hat{K})$,

$$|\hat{\chi} - I_{\hat{K}} \hat{\chi}|_{W^{m,q}(\hat{K})} \leq C \, |\hat{\chi}|_{W^{k+1,p}(\hat{K})} . \tag{5.10}$$

We re-scale (5.10) using Lemma 5.11 and the estimates from (5.5).

$$
\begin{aligned}
|\chi - I_K\chi|_{W^{m,q}(K)} &\le c \left(\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} \right)^{1/q} \left\| A_K^\dagger \right\|^m |\hat{\chi} - I_{\hat{K}}\hat{\chi}|_{W^{m,q}(\hat{K})} \\
&\le c \left(\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} \right)^{1/q} \left\| A_K^\dagger \right\|^m |\hat{\chi}|_{W^{k+1,p}(\hat{K})} \\
&\le c \left(\sup_{\hat{x}\in\hat{K}} \sqrt{g(\hat{x})} \right)^{1/q-1/p} \left\| A_K^\dagger \right\|^m \|A_K\|^{k+1} |\chi|_{W^{k+1,p}(K)} \\
&\le c \operatorname{meas}(K)^{1/q-1/p} \frac{h_K^{k+1}}{\rho_K^m} |\chi|_{W^{k+1,p}(K)} .
\end{aligned}
$$

In the final line we have used (5.5c). □

We will next bring together several surface finite elements in order to define Γ_h as a collection of finite element domains.

Definition 5.13. A *discrete hypersurface* is a set Γ_h equipped with an an *admissible subdivision* $\mathscr{T}_h$ consisting of surface finite element domains such that $\bigcup_{K\in\mathscr{T}_h} K = \Gamma_h$, $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$ for $K_1, K_2 \in \mathscr{T}_h$ with $K_1 \neq K_2$, and the constant C_K is uniformly bounded away from 1:

$$
\max_{K\in\mathscr{T}_h} C_K \le c < 1.
$$

We denote by h the maximum subdivision diameter:

$$
h := \max_{K\in\mathscr{T}_h} h_K. \tag{5.11}
$$

Remark 5.14. We do not impose any global assumptions on the connectivity or smoothness of Γ_h at this stage. However, in such cases there may not be an underling smooth surface.

We now restrict to Lagrangian finite elements over polygonal reference finite element. Here we assume that the degrees of freedom for each element (K, P, Σ) are given by

$$
\Sigma = \{\chi \mapsto \chi(a) : a \in \mathcal{N}^K\},
$$

where $\mathcal{N}^K$ is a finite set of nodes in K. We call $\mathcal{N}^K$ the set of *Lagrange nodes* of K. This restriction avoids difficulties in defining the edge of elements and how to effectively bring elements together to form a global finite element space. Extensions to other element types such as Hermite elements are left to future work.

Let Γ_h be a discrete hypersurface equipped with an admissible subdivision $\mathscr{T}_h$ such that each set $K \in \mathscr{T}_h$ is an element domain for a surface finite element (K, P^K, Σ^K) parameterised over the same polygonal reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$. We say that $\kappa \subset K$ is a *facet* if κ is the image of a boundary facet of $\hat{K}$. We say that $\mathscr{T}_h$ is a *conforming* subdivision of Γ_h if any facet of an element domain K is either a facet of another element domain $K' \in \mathscr{T}_h$, in which case we say K and K' are *adjacent*, or a portion of the boundary $\partial\Gamma_h$ (if such a boundary exists).

We orient a discrete hypersurface which is equipped with a conforming subdivision by choosing a particular sign to the element-wise definition of normal. We restrict that the induced orientation of the intersection of adjacent element domains do are opposite. For example, for a simplex reference element, the vertices in facets between two elements should be ordered oppositely in each element.

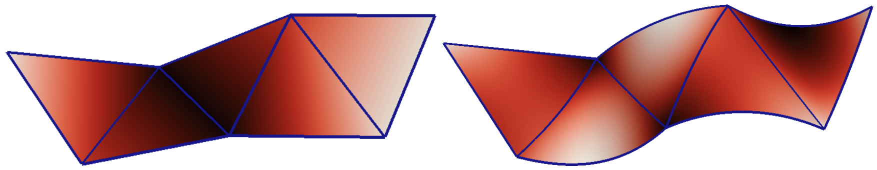

FIGURE 2. Examples of finite element functions. Left shows a piecewise linear function over a collection of affine finite elements and right shows a piecewise quadratic function over a collection of isoparametric (quadratic) finite elements.

Finally, the set of degrees of freedom of adjacent surface finite elements will be related as follows. Let (K,P,Σ) and (K',P',Σ') be two surface finite elements such that K and K' are adjacent with $\Sigma = \{\chi \mapsto \chi(a), a \in \mathcal{N}^K\}$ and $\Sigma' = \{\chi \mapsto \chi(a'), a' \in \mathcal{N}^{K'}\}$. Then, we have

$$\left(\bigcup_{a\in\mathcal{N}^K} a\right) \cap K' = \left(\bigcup_{a'\in\mathcal{N}^{K'}} a'\right) \cap K. \tag{5.12}$$

We denote the global set of Lagrange nodes by

$$\mathcal{N}_h = \bigcup_{K\in\mathscr{T}_h} \mathcal{N}^K.$$

For each $a \in \mathcal{N}_h$, let $\mathscr{T}(a) \subset \mathscr{T}_h$ be the local neighbourhood of elements for which $a \in \mathcal{N}^K$.

Definition 5.15 (Surface finite element space). Let Γ_h be a discrete hypersurface equipped with a conforming subdivision $\mathscr{T}_h$ with each domain K equipped with a surface finite element (K,P^K,Σ^K) which satisfy (5.12). A *surface finite element space* is a (generally proper) subset of the product space $\prod_{K\in\mathscr{T}_h} P^K$ given by

$$\mathcal{V}_h := \Bigg\{ \chi_h = (\chi_K)_{K\in\mathscr{T}_h} \in \prod_{K\in\mathscr{T}_h} P^K : \\ \chi_K(a) = \chi_{K'}(a), \text{ for all } K,K' \in \mathscr{T}(a), \text{ for all } a \in \mathcal{N}_h \Bigg\}$$

The surface finite element space is determined by the *global degrees of freedom*

$$\Sigma_h = \{\chi_h \mapsto \chi_h(a) : a \in \mathcal{N}_h\}.$$

In this definition, an element $\chi_h \in \mathcal{V}_h$ is not, in general a "function" defined over $\bar{\Gamma}_h$, since we do not necessarily have a good definition of χ_h over element boundaries. In the case of the usual embedding $P \subset C^0(K)$, we have from the assumption (5.12), that the "functions" $\chi_h \in \mathcal{V}_h$ are continuous at the nodes $\mathcal{N}_h$.

If it happens, however, that for each element $\chi_h \in \mathcal{V}_h$, the restrictions χ_K and $\chi_{K'}$ coincide along the common face of any adjacent elements K and K', then the function χ_h can be identified with a function defined over the set $\bar{\Gamma}_h$. In this case, we call the elements $\chi_h \in \mathcal{V}_h$ *surface finite element functions*. Examples of surface finite element functions are shown in Figure 2.

We enumerate the nodes so that $\mathcal{N}_h = \{a_i\}_{i=1}^N$ and take $\{\chi_i\}_{i=1}^N$ to be the basis of $\mathcal{V}_h$ dual to Σ_h. Since, we have a finite basis of $\mathcal{V}_h$ we note that we can identify any $\chi_h \in \mathcal{V}_h$

with a vector $\alpha \in \mathbb{R}^N$ so that

$$\chi_h(x) = \sum_{i=1}^N \alpha_i \chi_i(x) \qquad \text{for } x \in \Gamma_h.$$

Let $\mathcal{V}_h$ be a finite element space over a subdivision $\mathscr{T}_h$ consisting of θ-surface finite elements. Then, we define discrete broken Sobolev norms $\|\cdot\|_{W^{m,p}(\mathscr{T}_h)}$ for $0 \le m \le \theta$, $p \in [1,\infty]$ by

$$\|\chi_h\|_{W^{m,p}(\mathscr{T}_h)} := \begin{cases} \left(\sum_{K \in \mathscr{T}_h} \|\chi_h\|^p_{W^{m,p}(K)} \right)^{1/p} & p < \infty \\ \max_{K \in \mathscr{T}_h} \|\chi_h\|_{W^{m,\infty}(K)} & p = \infty. \end{cases} \tag{5.13}$$

If η is a functions on Γ_h for which all $\sigma_i(\eta)$, $1 \le i \le N$, is well defined (in case of Lagrangian finite elements, $\eta \in C(\Gamma_h)$ suffices), then we can define a *global interpolant* $I_h\eta$ by

$$I_h \eta = \sum_{i=1}^N \sigma_i(\eta) \chi_i.$$

Note that our construction implies that

$$(I_h\eta)|_K = I_K \eta|_K \text{ for all } K \in \mathscr{T}_h,$$

and $I_h \chi_h = \chi_h$ for all $\chi_h \in \mathcal{V}_h$.

In order to prove estimates on the global interpolant, we will first define two further properties of our subdivision $\mathscr{T}_h$.

Definition 5.16 (Regular and quasi-uniform subdivisions). For $h \in (0,h_0)$, let Γ_h be a discrete hypersurface equipped with a conforming subdivision $\mathscr{T}_h$. The family is said to be non-degenerate or *regular* if there exists $\rho > 0$ such that for all $K \in \mathscr{T}_h$ and all $h \in (0,h_0)$,

$$\rho_K \ge \rho h_K.$$

A regular family is said to be *quasi-uniform* if there exists $\rho > 0$ such that

$$\min\{\rho_K : K \in \mathscr{T}_h\} \ge \rho h \quad \text{ for all } h \in (0,h_0).$$

We note that for a regular subdivision there exists a constant $c > 0$ depending on the global quantities $\hat\rho, \hat h$ and ρ

$$\|A_K\| \le c h_K < ch \qquad \text{and} \qquad \left\| A_K^\dagger \right\| \le c h_K^{-1},$$

and for a quasi-uniform subdivision there exists a constant $c > 0$ depending on the global quantities $\hat\rho, \hat h$ and ρ

$$\|A_K\| \le ch \qquad \text{and} \qquad \left\| A_K^\dagger \right\| \le c h^{-1}.$$

Theorem 5.17 (Global interpolation estimates)**.** *For $h \in (0,h_0)$, let Γ_h be a discrete hypersurface equipped with a conforming regular subdivision $\mathscr{T}_h$. Let each $K \in \mathscr{T}_h$ be equipped with a θ-surface finite element (K, P^K, Σ^K) parameterised over a reference finite element $(\hat K, \hat P, \hat \Sigma)$ which satisfies the assumptions of Theorem 5.12 for some m,k,p,q. Then there exists a constant $C = C(\hat K, \hat P, \hat\Sigma, \rho)$ such that for all functions $\eta \in W^{k+1,p}(\mathscr{T}_h) \cap C^0(\Gamma_h)$,*

$$\|\eta - I_h\eta\|_{W^{m,q}(\mathscr{T}_h)} \le C h^{k+1-m} \|\eta\|_{W^{k+1,p}(\mathscr{T}_h)}. \tag{5.14}$$

Proof. The proof follows by piecing together Theorem 5.12 using the fact that $\mathscr{T}_h$ is quasi-uniform. □

Remark 5.18. In this work we will only use this result where the discrete hypersurface Γ_h is smooth so that the broken norm coincides with the smooth norm. This will be made more precise in Section 5.3.

5.2. **Evolving surface finite elements.** We now derive the correct formulations to define an evolving surface finite element space which is part of a compatible pair (in the sense of Section 2). For each $h \in (0,h)$, we are given a family of discrete hypersurfaces $\{\Gamma_h(t)\}_{t\in[0,T]}$ and each equipped with a surface finite element space $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$. Furthermore, we are interested in under what assumptions does the compatibility hold independently of the mesh size h.

Definition 5.19 (Evolving surface finite element). Let $(K(t),P(t),\Sigma(t))_{t\in[0,T]}$ be a time dependent family of surface finite elements parametrised over a common reference element (i.e. a single surface finite element evolving in time). If the constant $C_K = \max_t C_{K(t)}$ is uniformly bounded away from 1,

$$C_K := \max_{t\in[0,T]} \sup_{\hat{x}\in\hat{K}} \left\| D\Phi(\hat{x},t)A_K^\dagger(t) \right\| < c < 1,$$

we say that $(K(t),P(t),\Sigma(t))_{t\in[0,T]}$ is an *evolving surface finite element*.

The evolution of the element domain is given by a flow $\Phi_t^K \colon K_0 := K(0) \to K(t)$ given by

$$F_{K(t)}(\hat{x}) = \Phi_t^K(F_{K_0}(\hat{x})) \qquad \text{for } \hat{x} \in \hat{K}.$$

We assume that $\Phi_{(\cdot)}^K \in C^2([0,T],C^1(K_0))$. We can use the flow to define an element velocity W_K of $K(t)$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^K(x) = W_K(\Phi_t^K(x),t) \qquad \text{for } x \in K_0, t \in [0,T].$$

The flow defines a family of linear homeomorphisms $\phi_t^K \colon P_0 \to P(t)$. In order to show that $(P(t),\phi_t^K)$ is a compatible pair, we introduce a new definition which ensures that an element does not become too distorted during its evolution. We say that an evolving surface finite element is *quasi-uniform*, if there exists $\rho_K > 0$ such that

$$\inf\{\rho_{K(t)} : t \in [0,T]\} \geq \rho_K h.$$

Lemma 5.20. *Let $(K(t),P(t),\Sigma(t))_{t\in[0,T]}$ be a quasi-uniform θ-evolving surface finite element and ϕ_t^K the family of linear homeomorphisms defined by the flow Φ_t^K. Then there exists constants $c_1,c_2 > 0$, which depend only on the reference element $\hat{K}$ and the constants C_K and ρ_K, such that for all $t \in [0,T]$ and all $\chi \in P_0$ that $\chi \in W^{k,p}(K_0)$ if and only if $\phi_t^K\chi \in W^{k,p}(K(t))$ and*

$$c_1 |\chi|_{W^{k,p}(K_0)} \leq \left|\phi_t^K\chi\right|_{W^{k,p}(K(t))} \leq c_2 |\chi|_{W^{k,p}(K_0)}. \tag{5.15}$$

Proof. From Lemmas 5.11 and (5.5), we have

$$|\chi|_{W^{m,p}(K_0)} \leq c \left(\frac{\operatorname{meas}(K(t))}{\operatorname{meas}(K_0)}\right)^{1/p} \left(\frac{h_{K(t)}}{\rho_{K_0}}\right)^m \left|\phi_t^K\chi\right|_{W^{m,p}(K(t))}$$

and

$$\left|\phi_t^K\chi\right|_{W^{m,p}(K(t))} \leq c \left(\frac{\operatorname{meas}(K_0)}{\operatorname{meas}(K(t))}\right)^{1/p} \left(\frac{h_{K_0}}{\rho_{K(t)}}\right)^m |\chi|_{W^{m,p}(K_0)}.$$

It can be easily seen that for a quasi-uniform evolving surface finite element that these constants only depend on allowed quantities. □

We can bring together a collection of evolving surface finite elements in order to define an evolving discrete hypersurface. For $t \in [0,T]$, let Γ_h be a family of discrete hypersurfaces each equipped with a conforming subdivision $\mathscr{T}_h(t)$ of $\Gamma_h(t)$ such that each domain $K(t) \in \mathscr{T}_h(t)$ is equipped with a flow $\Phi_t^K \in C^2(0,T;C^1(K_0))$. We call $\{\mathscr{T}_h(t)\}_{t\in[0,t]}$ an *evolving conforming subdivision* if

- the curvedness constant C_K is uniformly bounded away from 1:
$$\max_{K(t)} C_K \leq c < 1;$$
- let K_0 and K_0' be two domains in $\mathscr{T}_h(0)$ which are adjacent, with associated flows Φ_t^K and $\Phi_t^{K'}$, then $\Phi_t^K(x) = \Phi_t^{K'}(x)$ for all $x \in K_0 \cap K_0'$ and $t \in [0,T]$.

We say that an *evolving discrete hypersurface* is a family of discrete hypersurfaces $\{\Gamma_h(t)\}_{t\in[0,T]}$ equipped with an evolving conforming subdivision.

Definition 5.21. Let $\{\Gamma_h(t)\}_{t\in[0,T]}$ be an evolving discrete hypersurface equipped with an evolving conforming subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$. For $t \in [0,T]$, let $\mathcal{V}_h(t)$ be a surface finite element space over $\Gamma_h(t)$. If each $K(t) \in \mathscr{T}_h(t)$ is equipped with an evolving surface finite element $(K(t),P(t),\Sigma(t))_{t\in[0,T]}$ then we say $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ is an *evolving finite element space*.

We define a global discrete flow $\Phi_t^h \colon \Gamma_{h,0} \to \Gamma_h$ element-wise by
$$\Phi_t^h|_{K_0} := \Phi_t^K \qquad \text{for } K_0 \in \mathscr{T}_h(0).$$
Our assumptions imply that Φ_t^h is piecewise smooth. We also have a global discrete velocity W_h given by
$$W_h|_{K(t)} = W_K.$$
The flow Φ_t^h induces a family of linear homeomorphisms, which we denote $\phi_t^h \colon \mathcal{V}_h(0) \to \mathcal{V}_h(t)$ given by
$$\phi_t^h \chi_h := \chi_h \circ \Phi_t^h \qquad \text{for } \chi_h \in \mathcal{V}_h(0).$$
This is equivalent to restricting to χ_h to its element-wise definition and using the push-forward map ϕ_t^K.

For $h \in (0,h_0)$, let $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ be a family of evolving conforming subdivisions. We say that the family is *uniformly regular* if there exists $\rho > 0$ such that for all $h \in (0,h_0)$ and all times $t \in [0,T]$, we have
$$\rho_{K(t)} \geq \rho h_{K(t)} \qquad \text{for all } K(t) \in \mathscr{T}_h(t).$$
We say that the family is *uniformly quasi-uniform* if there exists $\rho > 0$ such that for all $h \in (0,h_0)$ and all times $t \in [0,T]$, we have
$$\min\{\rho_{K(t)} : K(t) \in \mathscr{T}_h(t)\} \geq \rho h.$$

Note that a uniformly quasi-uniform subdivision consists of element domains for quasi-uniform evolving surface finite elements.

Lemma 5.22. *For $h \in (0,h_0)$, let $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ be an evolving surface finite element space over a uniformly quasi-uniform evolving conforming subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ consisting of θ-surface finite elements and ϕ_t^h the push-forward map. Let $0 \leq m \leq \theta$, $p \in [0,\infty]$, Then for $\chi_h \in \mathcal{V}_h(0)$, $\chi_h \in W^{k,p}(\mathscr{T}_h(0))$ if and only if $\phi_t^h \chi_h \in W^{k,p}(\mathscr{T}_h(t))$ for all $t \in [0,T]$.*

Furthermore, there exists $c_1, c_2 > 0$ *independent of* $h \in (0, h_0)$ *and* $t \in [0,T]$ *such that for all* $\chi \in \mathcal{V}_h(t)$

$$c_1 \|\chi_h\|_{W^{k,p}(\mathscr{T}_h(0))} \leq \left\|\phi_t^h \chi_h\right\|_{W^{k,p}(\mathscr{T}_h(t))} \leq c_2 \|\chi_h\|_{W^{k,p}(\mathscr{T}_h(0))} \tag{5.16}$$

In particular, the pair $(\mathcal{V}_h(t), \phi_t^h)_{t\in[0,T]}$ *is compatible with respect to the broken Sobolev norm* $\|\cdot\|_{W^{k,p}(\mathscr{T}_h(t))}$.

Proof. We simply sum the element-wise result from Lemma 5.20. The constants are independent of h_K and ρ_K due to the uniform quasi-uniformity of $\{\mathscr{T}_h(t)\}$. □

Note that this result implies that the spaces $L^2_{\mathcal{V}_h}$ and $C^m_{\mathcal{V}_h}$ are well defined.

Together with the global domain $\{\Gamma_h(t)\}_{t\in[0,T]}$, and finite element space $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$, for each $t \in [0,T]$, we will write $\Sigma_h(t)$ for the set of global nodal variables. We will use the convention that

$$\Sigma_h(t) = \{\chi_h \mapsto \chi_h(a_i(t)) : 1 \leq i \leq N\},$$

where $a_i(t)$ is the trajectory of a vertex under the global flow Φ_t^h. We will denote by $\{\chi_i(\cdot,t) : 1 \leq i \leq N\}$ the global basis of finite element functions such that $\chi_i(a_j(t),t) = \delta_{ij}$ for $t \in [0,T]$ and all $i, j = 1, \ldots, N$. This implies that $\chi_i(\cdot,t) = \phi_t^h(\chi_i(\cdot,0))$.

5.3. **Lifted finite elements.** So far we have only defined evolving surface finite elements without relation to the continuous setting. In general, due to the curvature of the surface or its boundary, the computational domain $\Gamma_h(t)$ can only only be an approximation to $\Gamma(t)$ in a practical scheme. We will identify evolving surface finite elements on $\Gamma_h(t)$ with a corresponding curved evolving surface finite elements on $\Gamma(t)$. We call this process lifting. In the following error analysis, we will compare the smooth solution with the lift of the discrete solution. We will also require an inverse lift that transforms functions on the smooth domain to the computational domain.

Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a C^2-evolving hypersurface with velocity w of class C^2, and, for $h \in (0,h_0)$, let $\{\Gamma_h(t)\}_{t\in[0,T]}$ be an evolving discrete hypersurface equipped with a uniformly regular evolving conforming subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ and an evolving surface finite element space $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$.

Definition 5.23. Write $(K,P,\Sigma)_{t\in[0,T]}$ for a surface finite element in $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$. Let $\Lambda_K(\cdot,t) \colon K(t) \to \Gamma(t)$ be of class C^2 in time and C^1 in space and such that there exists $c_1, c_2 > 0$ such that

$$c_1 \leq \inf_{x\in K} \|D\Lambda_K(x)\| \leq \sup_{x\in K} \|D\Lambda_K(x)\| \leq c_2.$$

Then the triple $(K^\ell, P^\ell, \Sigma^\ell)$ given by

$$\begin{aligned}
K^\ell &:= \Lambda_K(K)\\
P^\ell &:= \{\chi^\ell = \chi \circ \Lambda_K^{-1} : \chi \in P\}\\
\Sigma^\ell &:= \{\sigma^\ell := \chi^\ell \mapsto \sigma(\chi) : \sigma \in \Sigma\},
\end{aligned}$$

is called a *lifted finite element*. We call $(K^\ell, P^\ell, \Sigma^\ell)$ the *lift* of (K,P,Σ) and Λ_K the *lifting map*.

Remark 5.24. The above definition is sufficient in order to describe the finite element scheme. For the error analysis of the scheme, a key estimate will bound the difference of $D\Lambda_K$ to the identity map.

Fix $t \in [0,T]$. Let each $K(t) \in \mathcal{T}_h(t)$ be associated with a lifted finite element $K^\ell(t)$. We call the set of all lifted element domains $\mathcal{T}_h^\ell(t)$:

$$\mathcal{T}_h^\ell(t) = \{K^\ell(t) : K(t) \in \mathcal{T}_h(t)\}.$$

We assume that this forms an evolving conforming subdivision of $\Gamma(t)$. In this case, we say that $\{\mathcal{T}_h^\ell(t)\}_{t\in[0,T]}$ is an *exact subdivision* of the evolving hypersurface $\{\Gamma(t)\}_{t\in[0,T]}$.

Given an evolving surface finite element space, we can define a lifted evolving surface finite element by

$$\mathcal{V}_h^\ell(t) := \{\chi_h^\ell = \chi_h \circ \Lambda_h^{-1}(\cdot,t) : \chi_h \in \mathcal{V}_h(t)\}.$$

We can also use Λ_h to define an inverse lift. Under the above assumptions, we have an exact subdivision $\{\mathcal{T}_h^\ell(t)\}_{t\in[0,T]}$ of $\{\Gamma(t)\}_{t\in[0,T]}$. For $\eta \in C(\Gamma(t))$, we define the inverse lift of η, which denote by $\eta^{-\ell}$ as

$$\eta^{-\ell}(x) := \eta \circ \Lambda_h^{-1}(x) \qquad \text{for } x \in \Gamma_h(t).$$

Proposition 5.25. *Let $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ be an evolving surface finite element space consisting of θ-surface finite element methods and $\{\Gamma(t)\}_{t\in[0,T]}$ a C^θ-evolving hypersurface. The lifted evolving surface finite element space $\{\mathcal{V}_h^\ell(t)\}_{t\in[0,T]}$ is an evolving surface finite element space. Let $0 \le m \le \theta$ and $1 \le p \le \infty$. There exists constants $c_1, c_2 > 0$, independent of h and $t \in [0,T]$ such that for all $\chi_h \in \mathcal{V}_h(t)$ we have*

$$c_1 \|\chi_h\|_{W^{m,p}(\mathcal{T}_h(t))} \le \left\|\chi_h^\ell\right\|_{W^{m,p}(\Gamma(t))} \le c_2 \|\chi_h\|_{W^{m,p}(\mathcal{T}_h(t))} \tag{5.17}$$

and for all $\eta \in C(\Gamma(t)) \cap W^{m,p}(\Gamma(t))$ we have

$$c_1 \left\|\eta^{-\ell}\right\|_{W^{m,p}(\mathcal{T}_h(t))} \le \|\eta\|_{W^{m,p}(\Gamma(t))} \le c_2 \left\|\eta^{-\ell}\right\|_{W^{m,p}(\mathcal{T}_h(t))}. \tag{5.18}$$

Proof. The proof follows directly from the boundedness assumption on the Jacobian of Λ_h. We note that for the values of k,p allowed in the assumptions of this theorem the broken Sobolev norm $\|\cdot\|_{W^{k,p}(\mathcal{T}_h^\ell(t))}$ is equal to the usual Sobolev norm $\|\cdot\|_{W^{k,p}(\Gamma(t))}$. □

We can also show a special interpolation estimate which interpolates smooth functions over the continuous surface into the lifted finite element space.

Theorem 5.26 (Global lifted interpolation theorem)**.** *For $h \in (0,h_0)$, let $\{\Gamma_h(t)\}_{t\in[0,T]}$ be an evolving discrete hypersurface equipped with a uniformly quasi-uniform, conforming evolving subdivision $\{\mathcal{T}_h(t)\}_{t\in[0,T]}$. Let each $K(t) \in \mathcal{T}_h(t)$ be equipped with a θ-evolving surface finite element $(K(t), P^K(t), \Sigma^K(t))_{t\in[0,T]}$ parametrised over a reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ which satisfies the assumptions of Theorem 5.12 for some m,k,p,q. Let $\mathcal{T}_h^\ell(t)$ be a lifted subdivision with Λ_h a C^θ-diffeomorphism. Let $\eta \in C(\Gamma)$ be a continuous function, then $I_h\eta \in \mathcal{V}_h^\ell(t)$ is well defined. Furthermore there exists a constant $C = C(\hat{K}, \hat{P}, \hat{\Sigma}, \rho)$ such that for all functions $\eta \in W^{k+1,p}(\Gamma(t))$,*

$$\|\eta - I_h\eta\|_{W^{m,q}(\mathcal{T}_h^\ell(t))} \le Ch^{k+1-m} \|\eta\|_{W^{k+1,p}(\mathcal{T}_h^\ell(t))}. \tag{5.19}$$

Proof. Since the map Λ_h is a smooth diffeomorphism, the associated lifted finite element $(K^\ell(t), P^\ell(t), \Sigma^\ell(t))$ is also a θ-surface finite element for each time $t \in [0,T]$. Thus we can apply Theorem 5.12 for each lifted finite element to get the result. □

Corollary 5.27. *Let the same assumptions as Theorem 5.26 hold. Fix $t \in [0,T]$. Let $\eta \in C(\Gamma(t))$ then the interpolant of η into $\mathcal{V}_h(t)$ denoted by $\widetilde{I}_h\eta$ is given by*

$$\widetilde{I}_h\eta = (I_h\eta)^{-\ell} = I_h(\eta^{-\ell}).$$

Furthermore, there exists a constant C such that for all functions $\eta \in W^{k+1,p}(\Gamma(t))$*,*

$$\left\| \eta^{-\ell} - \widetilde{I}_h \eta \right\|_{W^{m,q}(\mathscr{T}_h(t))} \leq C h^{k+1-m} \|\eta\|_{W^{k+1,p}(\mathscr{T}_h^{\ell}(t))}. \tag{5.20}$$

The construction of the lifting map Λ_h induces a different flow $\Phi_t^{\ell} \colon \Gamma_0 \to \Gamma(t)$ of the smooth hypersurface given by

$$\Phi_t^{\ell}(\cdot) = \Lambda_h \circ \Phi_t^h \circ \Lambda_h^{-1}.$$

We note that in general we have that Φ_t^{ℓ} is different to Φ_t, but each describes a different parametrisation of the same evolving surface. This flow defines a different discrete velocity w_h on $\{\Gamma(t)\}_{t\in[0,T]}$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^{\ell}(\cdot) = w_h(\Phi_t^{\ell}(\cdot),t).$$

Note that w and w_h define the same surface, so have the same normal components, however the tangential components may not agree.

The flow induces a natural linear homeomorphism $\phi_t^{\ell} \colon \mathcal{V}_h^{\ell}(0) \to \mathcal{V}_h^{\ell}(t)$ given by

$$\phi_t^{\ell}(v_h)(x,t) = v_h(\Phi_{-t}^{\ell}(x)) \qquad \text{for } x \in \Gamma(t), v_h \in \mathcal{V}_h^{\ell}(0).$$

Proposition 5.28. *The pairs* $(\mathcal{V}_h^{\ell}(t), \phi_t^{\ell})_{t\in[0,T]}$ *and* $(W^{k,p}(\Gamma(t)), \phi_t^{\ell})$ *are also compatible when equipped with the* $W^{k,p}(\Gamma(t))$*-norm.*

Proof. The proof follows from the definition of flow map Φ_t^{ℓ} and the equivalence of norms shown in Proposition 5.25. □

5.4. **Discrete material derivatives and transport formulae.** We construct a discrete material derivative and transport formulae. We are equipped with the definitions and notation to define a discrete material velocity, discrete material derivative and transport formulae on $\Gamma_h(t)$ and also on $\Gamma(t)$.

We recall that we have three different velocities defined so far.

- The smooth velocity w on $\{\Gamma(t)\}_{t\in[0,T]}$, associated with the flow $\{\Phi_t\}_{t\in[0,T]}$ and push forward map $\{\phi_t\}_{t\in[0,T]}$, which will be used to define the continuous problem.
- The discrete velocity W_h on $\{\Gamma_h(t)\}_{t\in[0,T]}$, associated with the flow $\{\Phi_t^h\}$ and push forward map $\{\phi_t^h\}$, which will be used to define the discrete problem.
- The discrete velocity w_h, associated with the flow $\{\Phi_t^{\ell}\}$ and push forward map $\{\phi_t^{\ell}\}$, defines a different evolution of the continuous surface $\{\Gamma(t)\}$. This velocity is used as an analytical tool to construct a transformation of the discrete problem to a conforming approximation of the continuous problem with a variational crime and is not used in practical computations.

The discrete push-forward map, $\{\phi_t^h\}$, is a compatible pair with the finite element space $\{\mathcal{V}_h(t)\}$. Therefore we can define a *discrete material derivative* for $\chi_h \in C^1_{\mathcal{V}_h}$ given by

$$\partial_h^{\bullet}\chi_h := \phi_t^h\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_{-t}^h\chi_h\right).$$

We have a second discrete push-forward map, $\{\phi_t^{\ell}\}$, which is a compatible pair with both the lifted finite element space $\{\mathcal{V}_h^{\ell}(t)\}$ and also $\{L^2(\Gamma(t))\}$. Therefore, we can define a second *discrete material derivative* for functions $\eta \in C^1_{L^2(\Gamma(\cdot))}$ given by

$$\partial_h^{\bullet}\eta := \phi_t^{\ell}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_{-t}^{\ell}\eta\right).$$

We can characterise these two material derivatives using the discrete velocities W_h and w_h:

$$\partial_h^\bullet \chi_h = \partial_t \widetilde{\chi}_h + W_h \cdot \nabla \widetilde{\chi}_h \qquad \text{for } \chi_h \in C^1_{\mathcal{V}_h}$$
$$\partial_h^\bullet \eta = \partial_t \widetilde{\eta} + w_h \cdot \nabla \widetilde{\eta} \qquad \text{for } \eta \in C^1_{L^2(\Gamma(\cdot))}.$$

We also can compute that for $\chi_h \in C^1_{\mathcal{V}_h}$ that

$$(\partial_h^\bullet \chi_h)^\ell = \left(\phi_t^h \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi_{-t}^h \chi_h \right) \right)^\ell = \phi_t^\ell \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi_{-t}^\ell \chi_h^\ell \right) = \partial_h^\bullet (\chi_h^\ell),$$

since for $\chi_{h,0} \in \mathcal{V}_{h,0}$

$$\begin{aligned} \left(\phi_t^h \chi_{h,0} \right)^\ell &= \left(\chi_{h,0}(\Phi_{-t}(\cdot)) \right)^\ell = \chi_{h,0}(\Phi_{-t} \circ \Lambda_h(\cdot,t)) \\ &= \chi_{h,0}^\ell(\Lambda_h^{-1}(\cdot,t) \circ \Phi_{-t} \circ \Lambda_h(\cdot,t)) = \phi_t^\ell \chi_{h,0}^\ell. \end{aligned}$$

As a result we can extend the definition of $\partial_h^\bullet$. For $\eta \in C^1_{L^2(\Gamma(\cdot))}$, we define

$$\partial_h^\bullet \eta^{-\ell} := (\partial_h^\bullet \eta)^{-\ell}.$$

An important part of this construction is that basis functions have zero material derivative:

Lemma 5.29. *Denote by $\{\chi_j \cdot, t)\}_{j=1}^N$ the set of global basis functions. Then $\chi_j \in C^2_{\mathcal{V}_h}$ and $\partial_h^\bullet \chi_j = 0$ for $1 \le j \le N$. Furthermore, any function $\chi_h \in C^1_{\mathcal{V}_h}$, which can be written as $\chi_h = \sum_{j=1}^N \alpha_i(t) \chi_j(t)$, satisfies*

$$\partial_h^\bullet \chi_h(x,t) = \sum_{i=1}^N \alpha_j'(t) \chi_j(\cdot,t) \quad \text{for } x \in \Gamma_h(t). \tag{5.21}$$

Proof. Fix one element $K(t) \in \mathscr{T}_h(t)$. Let $x \in K(t)$ and $x_0 \in K_0$ such that $\Phi_t^K(x) = x_0$. For any basis function we have

$$\chi_j(x,t) = (\phi_t^K \chi_j)(x_0) = \chi_j(x_0, 0) \qquad \text{for } t \in [0,T],$$

hence we have that

$$\partial_K^\bullet \chi_j(x,t) = \phi_t^K \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi_{-t}^K v(\Phi_t^K x_0, t) \right) = \phi_t^K \left(\frac{\mathrm{d}}{\mathrm{d}t} \chi_j(x_0, 0) \right) = 0.$$

Since the global basis functions are composed on element basis functions, we have shown the result. □

Remark 5.30. Note that if $\chi_h \in C^1_{\mathcal{V}_h}$, Lemma 5.29 implies that $\partial_h^\bullet \chi_h \in C^0_{\mathcal{V}_h}$.

We have transport theorem for integrals over $\Gamma_h(t)$ and $\Gamma(t)$ which we derive by applying element-wise formulae over $\{\mathscr{T}_h(t)\}_{t \in [0,T]}$ and $\{\mathscr{T}_h^\ell(t)\}_{t \in [0,T]}$.

Lemma 5.31. *For $t \in [0,T]$ and for each $K(t) \in \mathscr{T}_h(t)$, let $\mathscr{A}_K$ be a smooth diffusion tensor on $K(t)$, which maps the tangent space of $K(t)$ to itself, and $\mathscr{B}_K$ be a smooth tangential* *vector field on* $K(t)$ *for each* $K(t) \in \mathscr{T}_h(t)$ *and* $t \in [0,T]$. *Let* $\chi_h, \phi_h \in C^1_{\mathcal{V}_h}$ *then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \chi_h \,\mathrm{d}\sigma_h \\
&= \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \partial_h^\bullet \chi_h + \chi_h \nabla_{\Gamma_h}\cdot W_h \,\mathrm{d}\sigma_h
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{A}_K \nabla_K \chi_h \cdot \nabla_K \phi_h \,\mathrm{d}\sigma_h \\
&= \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{A}_K \nabla_K \partial_h^\bullet \chi_h \cdot \nabla_K \phi_h + \mathscr{A}_K \nabla_K \chi_h \cdot \nabla_K \partial_h^\bullet \phi_h \,\mathrm{d}\sigma_h \\
&\quad + \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathcal{B}_K(W_K, \mathscr{A}_K) \nabla_K \chi_h \cdot \nabla_K \chi_h \,\mathrm{d}\sigma_h
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{B}_h \chi_h \cdot \nabla_K \phi_h \,\mathrm{d}\sigma_h \\
&= \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{B}_h (\partial_h^\bullet \chi_h \cdot \nabla_K \phi_h + \chi_h \cdot \nabla_K \partial_h^\bullet \phi_h) \,\mathrm{d}\sigma_h \\
&\quad + \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathcal{B}_{\mathrm{adv},K}(W_K, \mathscr{B}_h) \chi_h \cdot \nabla_K \phi_h \,\mathrm{d}\sigma_h,
\end{aligned} \tag{5.24}$$

where $\mathcal{B}_K$ *and* $\mathcal{B}_{\mathrm{adv},K}$ *are given by*

$$\mathcal{B}_K(W_K, \mathscr{A}_K) = \partial_K^\bullet \mathscr{A}_K + \nabla_K \cdot W_K \mathscr{A}_K - 2D_h(W_h)$$

$$\mathcal{B}_{\mathrm{adv},K}(W_K, \mathscr{B}_K) = \partial_K^\bullet \mathscr{B}_K + \mathscr{B}_K \nabla_K \cdot W_K - \sum_{j=1}^{n+1} (\mathscr{B}_K)_j (\nabla_K)_j W_K.$$

and D_h *is the rate of deformation tensor*

$$D(w)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1} (\mathscr{A}_K)_{ik} (\nabla_K)_j (W_K)_j + (\mathscr{A}_K)_{jk} (\nabla_K)_k (W_K)_i \quad \textit{for } i,j = 1,\dots,n+1.$$

Proof. Simply apply the result of (4.10), (4.11) and (4.13) element-wise. □

Lemma 5.32. *The flow map* Φ_t^ℓ *induces a new transport formula on* $\{\Gamma(t)\}$. *For* $\eta \in C^1_{L^2(\Gamma(\cdot))}$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \eta \,\mathrm{d}\sigma = \sum_{K^\ell(t)\in\mathscr{T}_h^\ell(t)} \int_{K^\ell(t)} \partial_h^\bullet \eta + \eta \nabla_\Gamma \cdot w_h \,\mathrm{d}\sigma. \tag{5.25}$$

Furthermore, we have for $\eta, \varphi \in C^1_{H^1(\Gamma(\cdot))}$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathscr{A}\nabla_\Gamma \eta \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma \\
&= \sum_{K^\ell(t) \in \mathscr{T}_h^\ell(t)} \int_{K^\ell(t)} \mathscr{A}\left(\nabla_\Gamma \partial^\bullet_h \eta \cdot \nabla_\Gamma + \nabla_\Gamma \eta \cdot \nabla_\Gamma \varphi\right) + \mathcal{B}_{K^\ell}(w_K, \mathscr{A}) \nabla_\Gamma \eta \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathscr{B}\eta \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma \\
&= \sum_{K^\ell(t) \in \mathscr{T}_h^\ell(t)} \int_{K^\ell(t)} \mathscr{B}\left(\partial^\bullet_h \eta \cdot \nabla_\Gamma + \eta \cdot \nabla_\Gamma \varphi\right) + \mathcal{B}_{K^\ell,\mathrm{adv}}(w_K, \mathscr{A}) \eta \cdot \nabla_\Gamma \varphi \,\mathrm{d}\sigma,
\end{aligned}
\tag{5.27}
$$

where $\mathcal{B}_{K^\ell}$ and $\mathcal{B}_{K^\ell,\mathrm{adv}}$ are defined as in Lemma 5.31.

Proof. We can apply the results of (4.10), (4.11) and (4.13) for each lifted element. □

6. Application I: Partial differential equation on a closed surface

The study of finite element methods for partial differential equations posed on surfaces started with the influential study of Dziuk (1988). This work has been extended to a heat equation posed on evolving surfaces by Dziuk and Elliott (2007). In this work we consider a more general parabolic equation on surfaces and discretisations which cover the case of an arbitrary Lagrangian-Eulerian scheme (Elliott and Styles, 2012; Elliott and Venkataraman, 2015) and higher-order schemes (Heine, 2005; Demlow, 2009). The methods presented in this paper can be combined with different time stepping schemes (Dziuk, Lubich, and Mansor, 2011; Dziuk and Elliott, 2012; Lubich, Mansour, and Venkataraman, 2013) to provide a fully discrete scheme. A similar construction has been presented by Kovács (2016) independently of this work.

Throughout this section we fix $k \in \mathbb{N}$ to be the desired polynomial order of basis functions used in the finite element method.

6.1. Continuous problem.

We take our notation from Section 4.1. For each $t \in [0,T]$, let $\Gamma(t)$ be an n-dimensional compact, orientable C^{k+1}-hypersurface embedded in $\mathbb{R}^{n+1}$, the image of $\Gamma_0 = \Gamma(0)$ under the flow Φ_t. We denote by $\mathcal{H}(t) = L^2(\Gamma(t))$, $\mathcal{V}(t) = H^1(\Gamma(t))$. These spaces form a compatible pair with the linear family of homeomorphisms $\{\phi_t\}_{t\in[0,T]}$ (4.9) and $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))_{t\in[0,T]}$ form a Hilbert triple. We will also make use of the spaces $\mathcal{Z}_0(t) = H^2(\Gamma(t))$ and $\mathcal{Z}(t) = H^{k+1}(\Gamma(t))$.

We will assume that $\Gamma(t)$ is a C^{k+1}-hypersurface and that the velocity field $w(\cdot,t) \in C^2(\Gamma(t))$ for each $t \in [0,T]$. We will assume that the velocity field is uniformly bounded in the sense that there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \|\nabla_\Gamma \cdot w\|_{L^\infty(\Gamma(t))} < C.$$

We assume that $\Gamma(t)$ is described by a distance function d such that

$$d, d_t, d_{x_i}, d_{x_i,x_j} \in C^{k+1}(\mathcal{N}_T) \qquad \text{for } i,j, = 1,\dots,n+1.$$

We assume that for each $t \in [0,T]$, $\mathscr{A}(\cdot,t)$ is a $(n+1)\times(n+1)$ symmetric diffusion tensor which maps the tangent space of $\Gamma(t)$ at a point into itself and is uniformly positive definite on the tangent space: There exists $a_0 > 0$ such that for all $t \in [0,T]$

$$\mathscr{A}(\cdot,t)\xi \cdot \xi \geq a_0 |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^{n+1}, \xi \cdot \nu(\cdot,t) = 0.$$

We assume that $\mathscr{A} \in C^1(\mathscr{G}_T;\mathbb{R}^{(n+1)\times(n+1)})$ and we also have a smooth tangential vector field $\mathscr{B} \in C^1(\mathscr{G}_T;\mathbb{R}^{n+1})$ and smooth scalar field $\mathscr{C} \in C^1(\mathscr{G}_T)$.

We start by formulating (1.3) in weak form:

Problem 6.1. *Given $u_0 \in \mathcal{V}_0$, find $u \in L^2_{\mathcal{V}}$ with $\partial^\bullet u \in L^2_{\mathcal{H}}$, such that for almost every time $t \in (0,T)$ we have*

$$m(t;\partial^\bullet u,\varphi)+g(t;u,\varphi)+a(t;u,\varphi)=0 \qquad \text{for all } \varphi \in \mathcal{V}(t) \tag{6.1}$$
$$u(\cdot,0)=u_0,$$

where for $\eta,\varphi \in \mathcal{V}(t)$ we define

$$m(t;\eta,\varphi) := \int_{\Gamma(t)} \eta\varphi \,\mathrm{d}\sigma$$
$$g(t;\eta,\varphi) := \int_{\Gamma(t)} \eta\varphi\nabla_\Gamma\cdot w \,\mathrm{d}\sigma$$
$$a(t;\eta,\varphi) := \int_{\Gamma(t)} \mathscr{A}\nabla_\Gamma\eta\cdot\nabla_\Gamma\varphi + \mathscr{B}\eta\cdot\nabla_\Gamma\varphi + \mathscr{C}\eta\varphi \,\mathrm{d}\sigma.$$

We note that first from (4.10), that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t;\eta,\eta) = 2m(t;\partial^\bullet\eta,\eta)+g(t;\eta,\eta) \quad \text{for all } \eta \in C^1_{\mathcal{H}}. \tag{6.2}$$

We have a transport formula for a from (4.11) and (4.13):

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t;\eta,\varphi) = a(t;\partial^\bullet\eta,\varphi)+a(t;\eta,\partial^\bullet\varphi)+b(t;\eta,\varphi) \qquad \text{for all } \eta,\varphi \in C^1_{\mathcal{V}}, \tag{6.3}$$

with

$$b(t;\eta,\varphi) = \int_{\Gamma(t)} \mathcal{B}(w,\mathscr{A})\nabla_\Gamma\eta\cdot\nabla_\Gamma\eta + \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B})\eta\cdot\nabla_\Gamma\varphi + (\partial^\bullet\mathscr{C}+\mathscr{C}\nabla_\Gamma\cdot w)\eta\varphi\,\mathrm{d}\sigma$$

where

$$\mathcal{B}(w,\mathscr{A}) = \partial^\bullet\mathscr{A} + \nabla_\Gamma\cdot w\mathscr{A} + D(w,\mathscr{A})$$
$$\mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}) = \partial^\bullet\mathscr{B} + \nabla_\Gamma\cdot w\mathscr{B} + \sum_{j=1}^{n+1}\mathscr{B}_j\underline{D}_j w.$$

Theorem 6.2. *There exists a unique solution u to (6.1) which satisfies*

$$\int_0^T \|u\|^2_{\mathcal{V}(t)} + \|\partial^\bullet u\|^2_{\mathcal{H}(t)}\,\mathrm{d}t \le c\,\|u_0\|^2_{\mathcal{V}_0}. \tag{6.4}$$

Proof. We simply check the assumptions required for Theorem 2.9. The assumptions (M1) and (M2) follow simply since $m(t;\cdot,\cdot)$ is the $\mathcal{H}(t)=L^2(\Gamma(t))$-inner product. (G1) holds from (4.10) and (2.2) from the assumption that $\|\nabla_\Gamma\cdot w\|_{L^\infty(\mathscr{G}_T)}$ is bounded. The bilinear form $a(t;\cdot,\cdot)$ is differentiable in time, hence measurable (A1). For the coercivity of $a(t;\cdot,\cdot)$ we have from the uniform positive definiteness of $\mathscr{A}$

$$\begin{aligned}
a_0\int_{\Gamma(t)}|\nabla_\Gamma\eta|^2\,\mathrm{d}\sigma &\le \int_{\Gamma(t)}\mathscr{A}\nabla_\Gamma\eta\cdot\nabla_\Gamma\eta\,\mathrm{d}\sigma\\
&= a(t;\eta,\eta) - \int_{\Gamma(t)}\mathscr{B}\cdot\eta\nabla_\Gamma\eta + \mathscr{C}\eta^2\,\mathrm{d}\sigma\\
&\le a(t;\eta,\eta) + \left(\frac{\|\mathscr{B}\|^2_{L^\infty(\mathscr{G}_T)}}{2a_0} + \|\mathscr{C}\|_{L^\infty(\mathscr{G}_T)}\right)\int_{\Gamma(t)}\eta^2\,\mathrm{d}\sigma + \frac{a_0}{2}\int_{\Gamma(t)}|\nabla_\Gamma\eta|^2\,\mathrm{d}\sigma.
\end{aligned}$$

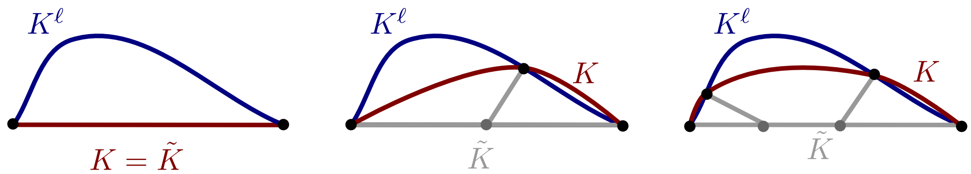

FIGURE 3. Examples of construction of an isoparametric surface finite element for $k = 1$ (left), $k = 2$ (centre), $k = 3$ (right). The Lagrange nodes $\widetilde{a}_i$ are shown in grey on $\widetilde{K}$ which are lifted to $K \subset \Gamma_{h,0}$ (red) to the Lagrange nodes $\widetilde{a}_i$ (black) which lie on the smooth surface $K \subset \Gamma_0$ (blue).

Hence, we have that

$$a(t;\eta,\eta) \geq \frac{a_0}{2} \|\eta\|^2_{\mathcal{V}(t)} - \left(\frac{\|\mathscr{B}\|^2_{L^\infty(\mathscr{G}_T)}}{2a_0} + \|\mathscr{C}\|_{L^\infty(\mathscr{G}_T)} \right) \|\eta\|^2_{\mathcal{H}(t)}.$$

This shows (A2). The smoothness of $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ imply that a is bounded (A3). The existence of the bilinear form b (B1) follows from (6.3) and the bound (B2) from the smoothness of $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$. □

6.2. **Finite element method.** The first stage of our finite element method is to define the approximate computational domain $\{\Gamma_h(t)\}$. We do this by construction an iso-parametric approximation of Γ_0 which is then pushed-forwards under an approximation to the flow Φ_t. The result will be that the Langrange points of $\Gamma_h(t)$ lie on the surface $\Gamma(t)$ for all times and evolving according to the velocity w. In this sense, $\Gamma_h(t)$ can be considered as an interpolation of $\Gamma(t)$.

Let $\widetilde{\Gamma}_{h,0}$ be a polyhedral approximation of Γ_0 equipped with a quasi-uniform, conforming subdivision $\widetilde{\mathscr{T}}_{h,0}$ (see Section 5.2 for details). We restrict that the vertices of $\widetilde{\Gamma}_{h,0}$ lie on the surface Γ_0 and denote by $\widetilde{h}_0$ the maximum mesh diameter on $\widetilde{\Gamma}_{h,0}$. If $\widetilde{h}_0$ is sufficiently small, the normal projection operator (4.2), $p(\cdot,0)$, is a smooth bijection from $\widetilde{\Gamma}_{h,0}$ onto Γ_0.

We equip each $\widetilde{K} \in \widetilde{\mathscr{T}}_{h,0}$ with a Lagrangian (affine) surface finite element $(\widetilde{K},\widetilde{P},\widetilde{\Sigma})$ of order k with $\widetilde{\Sigma}$ given by evaluation at the points $\{\widetilde{a}_i\}_{i=1}^{N_K} \subset \widetilde{K}$. Note that the vertices of $\widetilde{K}$ lie on Γ_0 but the other Lagrange points may not. We write $\widetilde{I}$ for the local interpolation operator over $(\widetilde{K},\widetilde{P},\widetilde{\Sigma})$ and lift the affine surface finite element onto a curved element (K,P^K,Σ^K) given by

$$\begin{aligned}
K &:= \{\widetilde{I}p(\widetilde{x},0) : \widetilde{x} \in \widetilde{K}\} \\
P &:= \{x \mapsto \widetilde{\chi}(\widetilde{x}) : \widetilde{I}p(\widetilde{x},0) = x \in K, \widetilde{\chi} \in \widetilde{P}\} \\
\Sigma &:= \{\chi \mapsto \chi(\widetilde{I}p(\widetilde{a}_i,0)) : 1 \leq i \leq N_K\}.
\end{aligned}$$

An example is given for $k = 1,2,3$ in Figure 3. We call the union of all elements constructed in this way $\mathscr{T}_{h,0}$ and call the union of elements domains $\Gamma_{h,0}$. Finally, we call $\{a_i\}_{i=1}^N$ given by $a_i = \widetilde{I}p(\widetilde{a}_i,0)$ the Lagrange nodes of $\Gamma_{h,0}$. We note that our construction implies that $a_i \in \Gamma_0$ for $1 \leq i \leq N$.

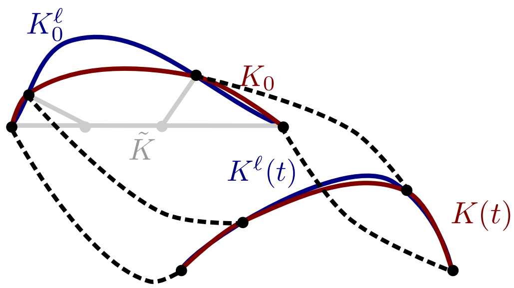

FIGURE 4. Examples of construction of an isoparametric evolving surface finite element for $k = 3$. The Lagrange nodes $a_i(t)$ follow the dashed black trajectories from the initial element $K_0 \subset \Gamma_0$ to a element $K(t) \subset \Gamma(t)$.

To complete the construction, for $K_0 \in \mathscr{T}_{h,0}$, we define the discrete domain $K(t)$ by the discrete flow $\Phi_t^K \colon K_0 \to K(t)$ defined element-wise by

$$\Phi_t^h|_{K_0} = \Phi_t^K := I_{K_0}[\Phi_t(p(\cdot,0))] \qquad \text{for all } K_0 \in \mathscr{T}_{h,0},$$

which is a bijection onto its image and we denote its inverse by Φ_{-t}^K. An example is shown in Figure 4. Since an iso-parametric element is defined by the location of its Lagrange points, we can consider that our evolving surface is defined by the initial location of the Lagrange points $\{a_i\}_{i=1}^N$ and their push forward under the smooth flow map $a_i(t) = \Phi_t(a_i)$ for $i = 1, \dots, N$.

Given a surface finite element (K_0, P_0, Σ_0), we define a surface finite element $(K(t), P^K(t), \Sigma^K(t))$ by

$$\begin{aligned} K(t) &= \Phi_t^K(K_0) \\ P^K(t) &= \{\chi(\Phi_{-t}^K) : \chi \in P_0\} \\ \Sigma^K(t) &= \{\chi \mapsto \chi(a_i(t)) : a_i(t) = \Phi_t^K(a_i), 1 \le i \le N_K\}. \end{aligned}$$

We call the union of such elements $\mathscr{T}_h(t)$ and we define $\Gamma_h(t)$ as

$$\Gamma_h(t) := \bigcup_{K(t) \in \mathscr{T}_h(t)} K(t),$$

and write a global discrete flow map $\Phi_t^h \colon \Gamma_{h,0} \to \Gamma_h(t)$ defined element-wise by $\Phi_t^h|_{K_0} := \Phi_t^K$. We define a global finite element space by

$$\mathcal{V}_h(t) := \big\{\phi_h \in C^0(\Gamma_h(t)) : \phi_h|_{K(t)} \in P^K(t) \text{ for all } K(t) \in \mathscr{T}_h(t)\big\}.$$

Remark 6.3. This construction is a generalisation of the construction of Dziuk and Elliott (2007). Indeed, in the case that we wish to consider affine finite elements, it is worth noting that $\widetilde{I}p(\widetilde{x}, 0) = \widetilde{x}$ for $\widetilde{x} \in \widetilde{K}$ and $K = \widetilde{K}$.

We will assume that this construction results in a uniformly quasi-uniform evolving subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$. It is clear that our construction maintains the conformity of the initial base triangulation $\widetilde{\mathscr{T}_{h,0}}$.

Proposition 6.4. *The above construction defines a uniformly regular evolving conforming subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ of $\{\Gamma_h(t)\}_{t\in[0,T]}$ and the evolving surface finite element space $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ consists of k-surface finite elements.*

Proof. Let $K(t)$ be a single element in $\mathscr{T}_h(t)$. We can write the element parametrisation $F_{K(t)}$ as

$$\begin{aligned} F_{K(t)}(\hat{x}) &= \Phi^t_K(F_{K_0}(\hat{x})) \\ &= \Phi^t_{\widetilde{K}}(F_{\widetilde{K}_0}(\hat{x})) + \big(\Phi^t_K(F_{K_0}(\hat{x})) - \Phi^t_{\widetilde{K}}(F_{\widetilde{K}_0}(\hat{x}))\big), \end{aligned}$$

where $\Phi^t_{\widetilde{K}}$ is the flow map from $\widetilde{K}_0 = F_{\widetilde{K}_0}(\hat{x})$ defined as the piecewise linear interpolant of Φ^t_K. We note that $\hat{x} \mapsto \Phi^t_{\widetilde{K}}(F_{\widetilde{K}_0}(\hat{x}))$ is linear so we define the splitting

$$F_{K(t)}(\hat{x}) = \underbrace{\Phi^t_{\widetilde{K}}(F_{\widetilde{K}_0}(\hat{x}))}_{=:\, A_{K(t)}\hat{x} + b_{K(t)}} + \underbrace{\big(\Phi^t_K(F_{K_0}(\hat{x})) - \Phi^t_{\widetilde{K}}(F_{\widetilde{K}_0}(\hat{x}))\big)}_{=:\, \Phi_{K(t)}(\hat{x})}.$$

Since $\Phi^t_K(F_{K_0}(\cdot))$ and $\Phi^t_K(F_{\widetilde{K}(F_{\widetilde{K}_0}(\cdot))})$ agree at the vertices of $\hat{K}$, we have that

$$\begin{aligned} \left| D_{\hat{x}} \left(\Phi^t_K(F_{K_0}(\cdot)) - \Phi^t_K(F_{\widetilde{K}(F_{\widetilde{K}_0}(\cdot))}) \right) \right| &= \left| D_{\hat{x}} \left(\Phi^t_K(F_{K_0}(\cdot)) - \widetilde{I}_1 \Phi^t_K(F_{K_0}(\cdot)) \right) \right| \\ &\leq c \left| \Phi^t_K(F_{K_0}(\cdot)) \right|_{W^{2,\infty}(\hat{K})} \\ &\leq c h_K^2 \left| \Phi^t_K \right|_{W^{2,\infty}(K_0)}. \end{aligned}$$

Here, we have used the notation $\widetilde{I}_1$ for piecewise linear interpolation over $\hat{K}$ and applied the Bramble-Hilbert Lemma (5.10) and the rescaling (5.6).

Hence we have

$$C_{K(t)} = \sup_{\hat{x} \in \hat{K}} \left\| D\Phi_K(\hat{x}) A_K^\dagger \right\| \leq c \frac{h_K^2}{\rho_K} \left| \Phi^t_K \right|_{W^{2,\infty}(K_0)}.$$

Since we have assumed that $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ is uniformly quasi-uniform, it remains to show that $|\Phi^t_K|_{W^{2,\infty}(K_0)}$ is uniformly bounded. However, this follows directly from the definition of Φ^t_K as an interpolation of $\Phi^t(p(\cdot,0))$ which is a smooth function. □

For each $t \in [0,T]$, and $h \in (0,h_0)$, we assume that $\mathscr{A}_h(t)$ is an element-wise smooth $(n+1)\times(n+1)$ symmetric diffusion tensor defined element-wise with $\mathscr{A}_h(t)|_{K(t)} = \mathscr{A}_K(t)$ for each $K(t) \in \mathscr{T}_h(t)$. We assume that $\mathscr{A}_K(t)$ maps the tangent space of $K(t)$ at a point into itself and is uniformly positive definite on the tangent space: There exists $a_0^h > 0$ such that for all $h \in (0,h_0)$, $t \in [0,T]$, and $K(t) \in \mathscr{T}_h(t)$

$$\mathscr{A}_K(\cdot,t)\xi \cdot \xi \geq a_0^h |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^{n+1}, \xi \cdot \nu_K(\cdot,t) = 0.$$

We assume we are also given a element-wise smooth tangential vector field $\mathscr{B}_h(t)$ (with $\mathscr{B}_h|_{K(t)} = \mathscr{B}_K$) and element-wise smooth scalar field $\mathscr{C}_h$ (with $\mathscr{C}_h|_{K(t)} = \mathscr{C}_K$). We assume that

$$\text{(6.5)} \qquad \sup_{t\in[0,T]} \sup_{K(t)\in\mathscr{T}_h(t)} \Big(\|\mathscr{A}_K\|_{C^1(K(t))} + \|\mathscr{B}_K\|_{C^1(K(t))} + \|\mathscr{C}_K\|_{C^1(K(t))} \Big) < C.$$

Example 6.5. Here we are thinking of the case that $\mathscr{A}_h = \mathscr{A}^{-\ell}$, $\mathscr{B}_h = \mathscr{B}^{-\ell}$ and $\mathscr{C}_h = \mathscr{C}^{-\ell}$.

We consider the following semi-discrete problem:

Problem 6.6. *Given $U_{h,0} \in \mathcal{V}_h(0)$, find $U_h \in C^1_{\mathcal{V}_h}$ such that for almost every $t \in (0,T)$,*

$$\text{(6.6)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} m_h(t;U_h,\phi_h) + a_h(t;U_h,\phi_h) = m_h(t;U_h,\partial_h^\bullet \phi_h) \qquad \textit{for all } \phi_h \in \mathcal{V}_h(t),$$

where

$$m_h(t;Z_h,\phi_h) = \int_{\Gamma_h(t)} Z_h\phi_h \,\mathrm{d}\sigma_h$$

$$a_h(t;Z_h,\phi_h) = \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{A}_h\nabla_K Z_h\cdot\nabla_K\phi_h + Z_h\nabla_K\phi_h\cdot\mathscr{B}_h + \mathscr{C}_h Z_h\phi_h \,\mathrm{d}\sigma_h.$$

6.3. **Stability.** To understand the stability properties of our method, we introduce some further terminology.

We equip $\mathcal{V}_h(t)$ with two norms:

$$\|\phi_h\|_{\mathcal{V}_h(t)} := \|\phi_h\|_{H^1(\mathscr{T}_h(t))} = \left(\sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} |\nabla_K\phi_h|^2 + \phi_h^2 \,\mathrm{d}\sigma_h\right)^{\frac{1}{2}}$$

$$\|\phi_h\|_{\mathcal{H}_h(t)} := \|\phi_h\|_{L^2(\Gamma_h(t))} = \left(\int_{\Gamma_h(t)} \phi_h^2 \,\mathrm{d}\sigma_h\right)^{\frac{1}{2}}.$$

We note that the assumption that $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ is uniformly quasi-uniform regular implies that $\{\mathcal{V}_h(t),\phi_t^h\}_{t\in[0,T]}$ is compatible when equipped with the $\mathcal{V}_h(t)$ or $\mathcal{H}(t)$-norms (Lemma 5.22).

We have transport formulae on the surface $\{\Gamma_h(t)\}$.

Lemma 6.7. *There exists a bilinear forms $g_h(t;\cdot,\cdot), b_h(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t)\to\mathbb{R}$ such that for all $Z_h,\phi_h\in C^1_{\mathcal{V}_h}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;Z_h,\phi_h) = m_h(t;\partial_h^\bullet Z_h,\phi_h) + m_h(t;Z_h,\partial_h^\bullet\phi_h) + g_h(t;Z_h,\phi_h) \tag{6.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_h(t;Z_h,\phi_h) = a_h(t;\partial_h^\bullet Z_h,\phi_h) + a_h(t;Z_h,\partial_h^\bullet\phi_h) + b_h(t;Z_h,\phi_h), \tag{6.8}$$

where

$$g_h(t;Z_h,\phi_h) = \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} Z_h\phi_h\nabla_K\cdot W_h \,\mathrm{d}\sigma_h \tag{6.9}$$

and

$$\begin{aligned} b_h(t;Z_h,\phi_h) = \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \Big(&\mathcal{B}_K(W_K,\mathscr{A}_K)\nabla_K Z_h\cdot\nabla_K\phi_h \\ &+ \mathcal{B}_{\mathrm{adv},K}(W_K,\mathscr{B}_K) Z_h\cdot\nabla_K\phi_h \\ &+ (\partial_K^\bullet\mathscr{C}_K + \mathscr{C}_K\nabla_K W_K) Z_h\phi_h\Big)\,\mathrm{d}\sigma_h. \end{aligned} \tag{6.10}$$

Furthermore, there exist a constant $c>0$ such that for all $t\in[0,T]$ and all $h\in(0,h_0)$ we have

$$|g_h(t;Z_h,\phi_h)| \le c\,\|Z_h\|_{\mathcal{H}_h(t)}\,\|\phi_h\|_{\mathcal{H}_h(t)} \qquad \text{for all } Z_h,\phi_h\in\mathcal{V}_h(t). \tag{6.11}$$

Proof. The transport theorem (5.22) directly gives (6.9) and additionally (5.23) and (5.24) give (6.10).

To see the boundedness property, we use that the above construction implies that

$$W_h(x,t) = \sum_{j=1}^N w(a_j(t),t)\chi_j(x,t) = (\widetilde{I}_h w)(x,t) \qquad \text{for } x\in\Gamma_h(t).$$

The interpolation bound (Corollary 5.27) then implies

$$\|\nabla_\Gamma W_h\|_{L^\infty(\Gamma_h(t))} \le c\,\|w\|_{W^{2,\infty}(\Gamma(t))}. \tag{6.12}$$

□

Theorem 6.8. *There exists a unique solution of the finite element scheme* (6.6). *The solution satisfies the stability bound:*

$$\sup_{t \in (0,T)} \|U_h\|^2_{\mathcal{H}_h(t)} + \int_0^T \|U_h\|^2_{\mathcal{V}_h(t)} \, \mathrm{d}t \leq c \left\| U_{h,0} \right\|^2_{\mathcal{H}_h(t)}. \tag{6.13}$$

Proof. The result is shown in the abstract setting in Theorem 3.3 so we are left to check the assumptions. The assumptions ($\mathrm{M}_h 1$) and ($\mathrm{M}_h 2$) follow since m_h is simply the $\mathcal{H}_h(t) = L^2(\Gamma_h(t))$ inner product. For ($\mathrm{G}_h 1$), we use (5.22) and the product rule $\partial_h^\bullet (v_h)^2 = 2 v_h \partial_h^\bullet v_h$. The bound ($\mathrm{G}_h 2$) is shown in Lemma 6.7. Finally, ($\mathrm{A}_h 1$), ($\mathrm{A}_h 2$) and ($\mathrm{A}_h 3$) follow from our assumptions on $\mathscr{A}_h, \mathscr{B}_h, \mathscr{C}_h$. □

6.4. **Error analysis.** Recalling the normal projection operator (4.2), we define the lifting operator $\Lambda_h(\cdot,t) \colon \Gamma_h(t) \to \Gamma(t)$ by

$$\Lambda_h(x,t) := p(x,t) \qquad \text{for } x \in \Gamma_h(t).$$

For $\phi_h \in \mathcal{V}_h(t)$, we denote its lift by $\phi_h^\ell(x)$ given by

$$\phi_h^\ell(x) = \phi_h((\Lambda_h)^{-1}(x,t)) \qquad \text{for } x \in \Gamma(t)$$

Lemma 6.9. *The lifting operator $\Lambda_h(\cdot,t) \colon \Gamma_h(t) \to \Gamma(t)$ is in $C^{k+1}(\Gamma_h(t))$ with the estimate:*

$$\sup_{t \in [0,T]} \|\Lambda_h(\cdot,t)\|_{W^{k+1,\infty}(\Gamma_h(t))} < C. \tag{6.14}$$

and furthermore,

$$\sup_{t \in [0,T]} \sup_{x \in \Gamma_h(t)} \|D\Lambda_h(x,t)\| \leq c, \qquad \sup_{t \in [0,T]} \sup_{y \in \Gamma(t)} \left\| D(\Lambda_h^{-1})(y,t) \right\| \leq c. \tag{6.15}$$

Proof. The first estimate follows from the smoothness of p (which follows from the smoothness of $\Gamma(t)$).

We compute directly that

$$\partial_{x_j}(p(x,t))_i = \delta_{ij} - \nu_i(x,t)\nu_j(x,t) - d(x,t)\mathcal{H}(x,t)_{ij}.$$

Then

$$\nabla_{\Gamma_h} \Lambda_h(x,t) = P_h(x,t) P(x,t) (\mathrm{Id} - d(x,t)\mathcal{H}(x,t)).$$ □

Lemma 6.10. *Let $\phi_h \in \mathcal{V}_h(t)$ and denote its lift by ϕ_h^ℓ. Then there exists constants $c_1, c_2 > 0$ such that*

$$c_1 \left\| \phi_h^\ell \right\|_{\mathcal{H}(t)} \leq \|\phi_h\|_{\mathcal{H}_h(t)} \leq c_2 \left\| \phi_h^\ell \right\|_{\mathcal{H}(t)} \tag{6.16}$$

$$c_1 \left\| \phi_h^\ell \right\|_{\mathcal{V}(t)} \leq \|\phi_h\|_{\mathcal{V}_h(t)} \leq c_2 \left\| \phi_h^\ell \right\|_{\mathcal{V}(t)}. \tag{6.17}$$

Proof. We apply Lemma 5.25 using the previous result (Lemma 6.9). □

We use the lift to define the space of lifted finite element functions $\mathcal{V}_h^\ell(t)$ by

$$\mathcal{V}_h^\ell(t) := \{ \phi_h^\ell : \phi_h \in \mathcal{V}_h(t) \}.$$

This space is equipped with the following approximation property

Lemma 6.11. *The interpolation operator $I_h \colon \mathcal{Z}_0(t) \to \mathcal{V}_h^\ell(t)$ is well defined and satisfies*

$$\|z - I_h z\|_{\mathcal{H}(t)} + h \|z - I_h z\|_{\mathcal{V}(t)} \leq c h^2 \|z\|_{\mathcal{Z}_0(t)} \qquad \text{for } z \in \mathcal{Z}_0(t) \tag{6.18}$$

$$\|z - I_h z\|_{\mathcal{H}(t)} + h \|z - I_h z\|_{\mathcal{V}(t)} \leq c h^{k+1} \|z\|_{\mathcal{Z}(t)} \qquad \text{for } z \in \mathcal{Z}(t). \tag{6.19}$$

Proof. We simply apply Theorem 5.26. □

The definition of lift gives us a lifted velocity w_h. Let $x = X(t) \in \Gamma_h(t)$ and $Y(t) = \Lambda_h(X(t),t)$. Then

$$w_h(p(x,t),t) = \frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \frac{\mathrm{d}}{\mathrm{d}t}p(X(t),t) = \nabla p(X(t),t)W_h(X(t),t) + p_t(X(t),t). \tag{6.20}$$

Then from the above calculation of $\partial_{x_j} p$ we have

$$w_h(p(x,t)) = (P(x,t) - d(x,t)\mathcal{H}(x,t))W_h(x,t) - d_t(x,t)\nu(x,t) - d(x,t)\nu_t(x,t). \tag{6.21}$$

Lemma 6.12. *We have the estimate:*

$$\|w - w_h\|_{L^\infty(\Gamma(t))} + h\,\|\nabla_\Gamma(w - w_h)\|_{L^\infty(\Gamma(t))} \leq ch^{k+1}. \tag{6.22}$$

Proof. We use the calculation of w_h from 6.21. Using the fact that $-d_t\nu + Pw = w$ we have for $x \in \Gamma_h(t)$ that

$$\begin{aligned} w(p(x,t),t) - w_h(p(x,t),t) = P(x,t)(w(p(x,t),t) - W_h(x,t)) \\ - d(x,t)(\mathcal{H}(x,t)W_h(x,t) + \nu_t(x,t)). \end{aligned}$$

Then the fact that W_h is an interpolant of w and the estimate (6.31) on d gives that

$$|w(p,\cdot) - w_h(p,\cdot)| \leq ch^{k+1}\,\|w\|_{W^{k+1}(\Gamma(t))}\,.$$

The gradient bound follows from a similar calculation since we have that

$$\begin{aligned} &(\nabla_\Gamma)_i\big(w(p(x,t),t) - w_h(p(x,t),t)\big) \\ &= ((\nabla_\Gamma)_i P(x,t))(w(p(x,t),t) - W_h(x,t)) + P(\nabla_\Gamma)_i(w(p(x,t),t) - W_h(x,t)) \\ &\quad + d(x,t)(\nabla_\Gamma)_i(\mathcal{H}(x,t)W_h(x,t) + \nu_t(x,t)). \end{aligned}$$

Here, we have used that $\nabla_\Gamma d = 0$. □

The lifting operator also defines transport formulae:

Lemma 6.13. *There exists bilinear forms $\widetilde{g}_h \colon \mathcal{H}(t) \times \mathcal{H}(t) \to \mathbb{R}$ and $\widetilde{b}_h \colon \mathcal{V}(t) \times \mathcal{V}(t)$ given by*

(6.23)
$$\widetilde{g}_h(t;\eta,\varphi) = \int_{\Gamma(t)} \eta\varphi\nabla_\Gamma \cdot w_h \,\mathrm{d}\sigma \qquad \text{for all } \eta,\varphi \in \mathcal{H}(t)$$

(6.24)
$$\begin{aligned} \widetilde{b}_h(t;\eta,\varphi) = \sum_{K^\ell(t) \in \mathscr{T}_h^\ell(t)} \int_{K^\ell(t)} \Big(&\mathcal{B}(w_h,\mathscr{A})\nabla_\Gamma\eta \cdot \nabla_\Gamma\varphi \\ &+ \mathcal{B}_{\mathrm{adv}}(w_h,\mathscr{B})\eta \cdot \nabla_\Gamma\varphi \\ &+ (\partial_h^\bullet \mathscr{C} + \mathscr{C}\nabla_\Gamma \cdot w_h)\eta\varphi \Big)\,\mathrm{d}\sigma \quad \text{for all } \eta,\varphi \in \mathcal{V}(t). \end{aligned}$$

These bilinear forms satisfy the following transport formulae on $\Gamma(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t;\eta,\varphi) = m(t;\partial_h^\bullet\eta,\varphi) + m(t;\eta,\partial_h^\bullet\varphi) + \widetilde{g}_h(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{H}} \tag{6.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t;\eta,\varphi) = a(t;\partial_h^\bullet\eta,\varphi) + a(t;\eta,\partial_h^\bullet\varphi) + \widetilde{b}_h(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{V}}. \tag{6.26}$$

Furthermore, the two new bilinear forms are uniformly bounded in the sense that there exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $h \in (0,h_0)$,

$$|\widetilde{g}_h(t;\eta,\varphi)| \leq c\,\|\eta\|_{\mathcal{H}(t)}\,\|\varphi\|_{\mathcal{H}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{H}(t) \tag{6.27}$$

$$\left|\widetilde{b}_h(t;\eta,\varphi)\right| \leq c\,\|\eta\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{V}(t). \tag{6.28}$$

Proof. The transport formulae are direct translations of Lemma 5.32. The bounds follow from the fact that $\|w_h\|_{W^{1,\infty}(\Gamma_h(t))}$ is bounded uniformly from Lemma 6.12. □

We also use the fact that Λ_h is invertible to define the inverse lift of a function $\varphi \in C(\Gamma(t))$ by

$$\varphi^{-\ell}(x) = \varphi(\Lambda_h(x,t)) \qquad \text{for } x \in \Gamma_h(t).$$

A Sobolev embedding tells us that $\mathcal{Z}_0(t) \subset C(\Gamma(t))$ so we can define the inverse lift of $\mathcal{Z}_0(t)$ to be

$$\mathcal{Z}_0^{-\ell}(t) := \{\varphi^{-\ell} : \varphi \in \mathcal{Z}_0(t)\}.$$

Theorem 6.14. *Let $\mathscr{A}_h = \mathscr{A}^{-\ell}, \mathscr{B}_h = \mathscr{B}^{-\ell}, \mathscr{C}_h = \mathscr{C}^{-\ell}$ and let $u \in L^2_{\mathcal{V}}$ be the solution of (6.1) which satisfies the further regularity requirement*

$$\sup_{t\in(0,T)} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|^2_{\mathcal{Z}(t)}\,\mathrm{d}t \leq C_u. \tag{6.29}$$

Let $U_h \in C^1_{\mathcal{V}_h}$ be the solution of (6.6) and denote its lift by $u_h = U_h^\ell$. Then we have the following error estimate

$$\sup_{t\in(0,T)} \|u-u_h\|^2_{\mathcal{H}(t)} + \int_0^T \|u-u_h\|^2_{\mathcal{V}(t)}\,\mathrm{d}t \leq c\left\|u-u_{h,0}\right\|^2_{\mathcal{H}(t)} + ch^{2k}C_u. \tag{6.30}$$

We begin by showing some basic geometric estimates.

Lemma 6.15. *Under the above smoothness assumptions, we have*

$$\sup_{t\in[0,T]} \max_{K(t)\in\mathscr{T}_h(t)} \|d\|_{L^\infty(K(t))} \leq ch^{k+1}. \tag{6.31}$$

$$\sup_{t\in[0,T]} \max_{K(t)\in\mathscr{T}_h(t)} \|\nu-\nu_K\|_{L^\infty(K(t))} \leq ch^{k} \tag{6.32}$$

$$\sup_{t\in[0,T]} \max_{K(t)\in\mathscr{T}_h(t)} \|H-H_K\|_{L^\infty(K(t))} \leq ch^{k-1}, \tag{6.33}$$

where $H_K := \nabla_K \nu_K$. Writing δ_h for the quotient between discrete and continuous surface measures so that $\mathrm{d}\sigma = \delta_h\,\mathrm{d}\sigma_h$, we have

$$\sup_{t\in[0,T]} \max_{K(t)\in\mathscr{T}_h(t)} \|1-\delta_h\|_{L^\infty(K(t))} \leq ch^{k+1}. \tag{6.34}$$

Proof. Consider a fixed time $t \in [0,T]$ and single element $K(t) \in \mathscr{T}_h(t)$ equipped with a finite element with nodes $\{a_i(t)\}_{i=1}^{N_K}$ Then d is a smooth function over $K(t)$ and $d(a_i(t)) = 0$ for $i = 1,\dots,N_K$ hence $I_K d = 0$. The interpolation estimate (Theorem 5.12) we have

$$\|d\|_{L^\infty(K(t))} + h\,\|\nabla_K d\|_{L^\infty(K(t))} + h^{-1}\left\|\nabla_K^2 d\right\|_{L^\infty(K(t))} \leq ch^{k+1}. \tag{6.35}$$

This implies the first result (6.31) and also since $P_h\nu = P_h\nabla d = \nabla_K d$ that

$$\|P_h\nu\|_{L^\infty(K(t))} \leq ch^k.$$

Then we have for $x \in K(t)$ that

$$\nu(x,t) - \nu_h(x,t) = P_h(x,t)\nu(x,t) + (\nu(x,t) - \nu_h(x,t)) \cdot \nu_h(x,t)\nu_h(x,t).$$

However, using the fact that ν is a unit vector field and $P_h\nu$ is orthogonal to ν_h we have

$$\begin{aligned}
|(\nu(x,t) - \nu_h(x,t)) \cdot \nu_h(x,t)| &= |(\nu(x,t) - P_h(x,t)\nu(x,t)) \cdot \nu_h(x,t) - 1| \\
&= ||\nu(x,t) - P_h(x,t)\nu(x,t)| - 1| \\
&= \left| \left(1 - |P_h(x,t)\nu(x,t)|^2\right)^{\frac{1}{2}} - 1 \right| \\
&= \frac{|P_h(x,t)\nu(x,t)|^2}{\left(1 - |P_h(x,t)\nu(x,t)|^2\right)^{\frac{1}{2}} + 1} \\
&\le |P_h(x,t)\nu(x,t)|^2 \le ch^{2k}.
\end{aligned}$$

Hence, we have

$$\|\nu - \nu_h\|_{L^\infty(K(t))} \le ch^k,$$

Similarly expanding $\nabla_K^2 d$ and using the fact that $H\nu = H_K\nu_K = 0$, we see that

$$\begin{aligned}
(H - H_K)_{ij} = -(H_K)_{ij}(1 - \nu_K \cdot \nu) + (H\nu_K)_j(\nu_K)_i + (\nu_K)_j \left[P_h(H\nu_K + H_K\nu)\right]_i \\
+ (\nabla_K^2 d)_{ij}.
\end{aligned}$$

Applying the interpolation estimate (6.35) and the previous estimate (6.32) shows the bound (6.33).

From (Demlow and Dziuk, 2008), we have that

$$\delta_h(x,t) = \nu(x,t) \cdot \nu_h(x,t) \prod_{i=1}^{n} (1 - d(x,t)\kappa_i(x,t)), \qquad \text{for } x \in K(t), \tag{6.36}$$

where $\kappa_i(x,t) = \dfrac{\kappa_i(p(x,t),t)}{1 + d(x,t)\kappa_i(p(x,t),t)}$ and $\kappa_i(p,\cdot)$, for $p \in \Gamma(t)$, is the ith principal curvature. The result (6.34) then follows by applying (6.31) and the estimate

$$|1 - \nu \cdot \nu_h| = |(\nu_h - \nu) \cdot \nu_h)| \le \|\nu_h - \nu\| \le ch^k. \qquad \square$$

Lemma 6.16.

$$\sup_{t \in [0,T]} \|\partial_h^\bullet d\|_{L^\infty(\Gamma_h(t))} \le ch^{k+1} \tag{6.37}$$

$$\sup_{t \in [0,T]} \|\partial_h^\bullet P\nu_h\|_{L^\infty(\Gamma_h(t))} \le ch^k \tag{6.38}$$

$$\sup_{t \in [0,T]} \|\partial_h^\bullet \delta_h\|_{L^\infty(\Gamma_h(t))} \le ch^{k+1}. \tag{6.39}$$

Proof. Again, we consider a fixed time $t \in [0,T]$ and a single element $K(t) \in \mathscr{T}_h(t)$. Following the same reasoning as (6.35) we know that $I_K \partial_h^\bullet d = 0$, hence

$$\|\partial_h^\bullet d\|_{L^\infty(K(t))} + h\|\nabla_K \partial_h^\bullet d\|_{L^\infty(K(t))} \le ch^{k+1}. \tag{6.40}$$

This immediately shows (6.37).

Next, we use (4.14) and write $\underline{D}_i^K$ for $(\nabla_K)_i$ to see that

$$\begin{aligned}
\partial_h^\bullet \big[P_h(x,t)\nu(x,t)\big]_i &= \partial_h^\bullet \big[(\nabla_K)_i d(x,t)\big]\big] \\
&= \underline{D}_i^K \partial_h^\bullet d(x,t) - \sum_{j=1}^{n+1} \underline{D}_i^K (W_h(x,t))_j \underline{D}_j^K d(x,t) \\
&\quad + \Big(\nabla_K (W_h(x,t)\cdot \nu_h(x,t)) \cdot \nabla_K d(x,t) \\
&\qquad - \sum_{j,l=1}^{n+1} (W_h)_j \underline{D}_l^K d(x,t) \underline{D}_l^K (\nu_h(x,t))_j \Big) (\nu_h)_i.
\end{aligned}$$

So that using the estimates (6.35) and (6.40) with the bound (6.12) on $\|W_h\|_{W^{1,\infty}(\Gamma_h(t))}$ and (6.33) gives that $\|\nabla_K \nu_h\|_{L^\infty(K(t))}$ is uniformly bounded, we infer that

$$\begin{aligned}
\|\partial_h^\bullet (P_h\nu)\|_{L^\infty(K(t))} &\le \|\partial_h^\bullet (\nabla_K d)\|_{L^\infty(K(t))} + c\,\|\nabla_K W_h\|_{L^\infty(K(t))}\,\|\nabla_K d\|_{L^\infty(K(t))} \\
&\quad + c\,\|W_h\|_{L^\infty(K(t))}\,\|\nabla_K d\|_{L^\infty(K(t))}\,\|\nabla_K \nu_h\|_{L^\infty(K(t))} \\
&\le ch^k.
\end{aligned}$$

Furthermore, for $x \in K(t)$

$$\begin{aligned}
\partial_h^\bullet \big[P(x,t)\nu_h(x,t)\big] &= \partial_h^\bullet \big[P(x,t)\big(\nu_h(x,t) - \nu(x,t)\big)\big] \\
&= (\partial_h^\bullet P(x,t))\big(\nu_h(x,t) - \nu(x,t)\big) + P(x,t)\partial_h^\bullet \big(\nu_h(x,t) - \nu(x,t)\big).
\end{aligned}$$

Using similar arguments to Lemma 6.15 we have

$$\begin{aligned}
\partial_h^\bullet \big[\nu(x,t) - \nu_h(x,t)\big] &= \partial_h^\bullet \big[P_h(x,t)\nu(x,t)\big] \\
&\quad + \partial_h^\bullet \left(\big[|\nu(x,t) - P_h(x,t)\nu(x,t)| - 1 \big] \nu_h(x,t) \right) \\
&= \partial_h^\bullet \big[P_h(x,t)\nu(x,t)\big] \\
&\quad - \frac{\partial_h^\bullet \big[P_h(x,t)\nu(x,t)\big]\cdot \big[P_h(x,t)\nu(x,t)\big]}{\Big(1 - |P_h(x,t)\nu(x,t)|^2\Big)^{\frac12}} \\
&\quad - \frac{|P_h(x,t)\nu(x,t)|^2}{\Big(1 + |P_h\nu(x,t)|^2\Big)^{\frac12} + 1}\,\partial_h^\bullet \nu_h(x,t).
\end{aligned}$$

Hence, we apply the bound on $\big\|\partial_h^\bullet (P_h\nu)\big\|_{L^\infty(K(t))}$ and $\|\nu - \nu_h\|_{L^\infty(K(t))}$ we infer that

$$\begin{aligned}
\|\partial_h^\bullet (P\nu_h)\|_{L^\infty(K(t))} &\le \|\partial_h^\bullet \nu\|_{L^\infty(K(t))}\,\|\nu - \nu_h\|_{L^\infty(K(t))} \\
&\quad + \|\partial_h^\bullet (P_h\nu)\|_{L^\infty(K(t))} + c\,\|\partial_h^\bullet (P_h\nu)\|_{L^\infty(K(t))}\,\|P_h\nu\|_{L^\infty(K(t))} \\
&\quad + \|P_h\nu\|^2_{L^\infty(K(t))}\,\|\partial_h^\bullet \nu_h\|_{L^\infty(K(t))} \\
&\le ch^k \left(\|\partial_h^\bullet \nu\|_{L^\infty(K(t))} + \|\partial_h^\bullet \nu_h\|_{L^\infty(K(t))} \right).
\end{aligned}$$

It remains to show that the final terms on the right hand side of this inequality are bounded independently of the choice of $K(t)$ and $t \in [0,T]$. First, we see that from (4.15) together with the fact that (6.33) gives that $\|\nabla_K \nu_h\|_{L^\infty(K(t))}$ is uniformly bounded, then

$$|\partial_h^\bullet \nu_h(x,t)| \le \|W_h\|_{W^{1,\infty}(K(t))}\,(1 + \|\nabla_K \nu_h(x,t)\|).$$

Second, from the definition of lifted material derivative and the velocities w and w we have

$$|\partial_h^\bullet v(x,t)| = |\partial_h^\bullet v(p(x,t),t)| \le |\partial^\bullet v(p(x,t),t)| + \big|\nabla_\Gamma v(p(x,t),t)\big(w(x,t) - w_h(x,t)\big)\big|$$
$$\le |\partial^\bullet v(p(x,t),t)| + |\mathcal{H}(p(x,t),t)|\, ch^{k+1}.$$

In the last line, we have applied Lemma 6.12. The fact that the right hand side is then bounded follows from the smoothness of $\Gamma(t)$ and w.

Finally, to show (6.39), we use (6.36) again together with the estimate (6.37) and the bounds on $\partial_h^\bullet v_h$ and $\partial_h^\bullet v$ from above. □

Let $V_h \in \mathcal{V}_h(t)$ and denote by $v_h = \mathcal{V}_h^\ell(t)$. A small calculation shows that we have

$$\nabla_{\Gamma_h} V_h(x,t) = P_h(x,t)(\mathrm{Id} - d(x,t)\mathcal{H}(x,t))\nabla_\Gamma v_h(p(x,t),t), \qquad \text{for } x \in \Gamma_h(t).$$

Then for $V_h, \phi_h \in \mathcal{V}_h(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h \in \mathcal{V}_h^\ell(t)$ we have

$$\begin{aligned}
&\int_{\Gamma_h(t)} \mathscr{A}_h \nabla_{\Gamma_h} V_h \cdot \nabla_{\Gamma_h} \phi_h \,\mathrm{d}\sigma_h \\
&\quad = \int_{\Gamma(t)} \left[\frac{1}{\delta_h}(\mathrm{Id} - d\mathcal{H})PP_hP\mathscr{A}_hPP_hP(\mathrm{Id} - d\mathcal{H})\right]^\ell \nabla_\Gamma v_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}\sigma \\
&\quad =: \int_{\Gamma(t)} \left[\mathcal{Q}_h^1\right]^\ell \nabla_\Gamma v_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}\sigma \\
&\int_{\Gamma_h(t)} \mathscr{B}_h V_h \cdot \nabla_{\Gamma_h} \phi_h \,\mathrm{d}\sigma_h = \int_{\Gamma(t)} \left[\frac{1}{\delta_h}(\mathrm{Id} - d\mathcal{H})PP_hP\mathscr{B}_h\right]^\ell v_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}\sigma_h \\
&\quad =: \int_{\Gamma(t)} \left[\mathcal{Q}_h^2\right]^\ell v_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}\sigma \\
&\int_{\Gamma_h(t)} \mathscr{C}_h V_h \phi_h \,\mathrm{d}\sigma_h = \int_{\Gamma(t)} \left[\frac{1}{\delta_h}\mathscr{C}_h\right]^\ell v_h \varphi_h \,\mathrm{d}\sigma =: \int_{\Gamma(t)} \left[\mathcal{Q}_h^3\right]^\ell v_h \varphi_h \,\mathrm{d}\sigma.
\end{aligned}$$

Lemma 6.17. *There exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $V_h, \phi_h \in \mathcal{V}_h(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t)$ we have*

(6.41a) $$|m(t; v_h, \varphi_h) - m_h(t; V_h, \phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$$

(6.41b) $$|a(t; v_h, \varphi_h) - a_h(t; V_h, \phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$$

(6.41c) $$|\widetilde{g}_h(t; v_h, \varphi_h) - g_h(t; V_h, \phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}$$

(6.41d) $$\left|\widetilde{b}_h(t; v_h, \varphi_h) - b_h(t; V_h, \phi_h)\right| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}.$$

Proof. For (6.41a), we apply (6.34):

$$|m(t; v_h, \varphi_h) - m_h(t; V_h, \phi_h)| = \left|\int_{\Gamma(t)} v_h \varphi_h \left(1 - \frac{1}{\delta_h^\ell}\right) \mathrm{d}\sigma\right|$$
$$\le ch^k \|v_h\|_{L^2(\Gamma(t))} \|\varphi_h\|_{L^2(\Gamma(t))}.$$

For (6.41b), we have that

(6.42) $$\begin{aligned}
&|a(t; v_h, \varphi_h) - a_h(t; V_h, \phi_h)| \\
&= \int_{\Gamma(t)} \Big\{ \left|\mathscr{A} - \left[\mathcal{Q}_h^1\right]^\ell\right| \nabla_\Gamma v_h \cdot \nabla_\Gamma \varphi_h + \left|\mathscr{B} - \left[\mathcal{Q}_h^2\right]^\ell\right| v_h \cdot \nabla_\Gamma \varphi_h \\
&\qquad + \left|\mathscr{C} - \left[\mathcal{Q}_h^3\right]^\ell\right| v_h \varphi_h \Big\} \,\mathrm{d}\sigma.
\end{aligned}$$

Then applying (6.31) and (6.34) we have

$$\begin{aligned}\mathscr{A}^{-\ell} - \mathcal{Q}_h^1 &= P\mathscr{A}_h P - \frac{1}{\delta_h}(\mathrm{Id} - d\mathcal{H})PP_hP\mathscr{A}_h PP_hP(\mathrm{Id} - d\mathcal{H}) \\ &= P\mathscr{A}_h P - PP_hP\mathscr{A}_h PP_hP + O(h^{k+1}) \\ &= \frac{1}{2}(P - PP_hP)\mathscr{A}_h(P + PP_hP) + \frac{1}{2}(P + PP_hP)\mathscr{A}_h(P - PP_hP) \\ &\quad + O(h^{k+1}).\end{aligned}$$

A small calculation shows that

$$P - PP_hP = (P\nu_h) \otimes (P\nu_h).$$

Hence, applying (6.32) we have

$$\left|\mathscr{A}^{-\ell} - \mathcal{Q}_h^1\right| \le c \left\|P(\nu_h - \nu)\right\|^2_{L^\infty(K(t))} + ch^{k+1} \le ch^{k+1}.$$

Similarly,

$$\begin{aligned}\left|\mathscr{B}^{-\ell} - \mathcal{Q}_h^2\right| &= \left|P\mathscr{B}_h - \frac{1}{\delta_h}(\mathrm{Id} - d\mathcal{H})PP_hP\mathscr{B}_h\right| \\ &\le c\left|P - PP_hP\right| + ch^{k+1} = c\left|P\nu_h\right|^2 + ch^{k+1} \le ch^{k+1},\end{aligned}$$

and finally,

$$\left|\mathscr{C}^{-\ell} - \mathcal{Q}_h^3\right| = \left|\mathscr{C}_h - \mathscr{C}_h \frac{1}{\delta_h}\right| \le c\left|1 - \frac{1}{\delta_h}\right| \le ch^{k+1}.$$

Combining these three estimates with (6.42) implies the result.

Next, for (6.41c), following (Ranner, 2013, Lemma 3.3.14) we rewrite $\widetilde{g}_h(t; v_h, \varphi_h)$ in two different ways. First, using (6.7)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t; V_h, \phi_h) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} V_h \phi_h \,\mathrm{d}\sigma_h \\ &= \int_{\Gamma_h(t)} \partial_h^\bullet V_h \phi_h + V_h \partial_h^\bullet \phi_h \,\mathrm{d}\sigma_h + g_h(t; V_h, \phi_h) \\ &= \int_{\Gamma(t)} (\partial_h^\bullet v_h \varphi_h + v_h \partial_h^\bullet \varphi_h) \frac{1}{\delta_h^\ell} \,\mathrm{d}\sigma + g_h(t; V_h, \phi_h).\end{aligned}$$

Secondly, using (6.25)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m_h(t; V_h, \phi_h) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} V_h \phi_h \,\mathrm{d}\sigma_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} v_h \varphi_h \frac{1}{\delta_h^\ell}\,\mathrm{d}\sigma \\ &= \int_{\Gamma(t)} (\partial_h^\bullet v_h \varphi_h + v_h \partial_h^\bullet \varphi_h) \frac{1}{\delta_h^\ell}\,\mathrm{d}\sigma \\ &\quad + \int_{\Gamma(t)} v_h \varphi_h \left(\partial_h^\bullet \frac{1}{\delta_h^\ell} + \left(\frac{1}{\delta_h^\ell} - 1\right)\nabla_\Gamma \cdot w_h\right)\mathrm{d}\sigma + \widetilde{g}_h(t; v_h, \varphi_h).\end{aligned}$$

Hence we have that by applying (6.39) and (6.34)

$$\begin{aligned}&\left|\widetilde{g}_h(t; v_h, \varphi_h) - g_h(t; V_h, \phi_h)\right| \\ &\le \left\|v_h\right\|_{L^2(\Gamma(t))}\left\|\varphi_h\right\|_{L^2(\Gamma(t))}\left(\left\|\partial_h^\bullet \frac{1}{\delta_h}\right\|_{L^\infty(\Gamma_h(t))} + \left\|\frac{1}{\delta_h} - 1\right\|_{L^\infty(\Gamma_h(t))}\right) \\ &\le ch^{k+1}\left\|v_h\right\|_{L^2(\Gamma(t))}\left\|\varphi_h\right\|_{L^2(\Gamma(t))}.\end{aligned}$$

We apply a similar idea to $\widetilde{b}_h(t;v_h,\varphi_h)$. First, using (6.8), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} a_h(t;V_h,\phi_h)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \mathscr{A}_h \nabla_{\Gamma_h} V_h\cdot\nabla_{\Gamma_h}\phi_h + \mathscr{B}_h V_h\cdot\nabla_{\Gamma_h}\phi_h + \mathscr{C}_h V_h\phi_h \,\mathrm{d}\sigma_h\\
&= \int_{\Gamma_h(t)} \mathscr{A}_h\Big(\nabla_{\Gamma_h}\partial_h^\bullet V_h\cdot\nabla_{\Gamma_h}\phi_h + \nabla_{\Gamma_h}V_h\cdot\nabla_{\Gamma_h}\partial_h^\bullet\phi_h\Big)\\
&\qquad + \mathscr{B}_h\Big(\partial_h^\bullet V_h\cdot\nabla_{\Gamma_h}\phi_h + V_h\cdot\nabla_{\Gamma_h}\partial_h^\bullet\phi_h\Big)\\
&\qquad + \mathscr{C}_h\Big(\partial_h^\bullet V_h\phi_h + \mathscr{C}_h V_h\partial_h^\bullet\phi_h\Big)\,\mathrm{d}\sigma_h + b_h(t;V_h,\phi_h)\\
&= \int_{\Gamma(t)} \big[\mathcal{Q}_h^1\big]^\ell\Big(\nabla_\Gamma\partial_h^\bullet v_h\cdot\nabla_\Gamma\varphi_h + \nabla_\Gamma v_h\cdot\nabla_\Gamma\partial_h^\bullet\varphi_h\Big)\\
&\qquad + \big[\mathcal{Q}_h^2\big]^\ell\Big(\partial_h^\bullet v_h\cdot\nabla_\Gamma\varphi_h + v_h\cdot\nabla_\Gamma\partial_h^\bullet\varphi_h\Big)\\
&\qquad + \big[\mathcal{Q}_h^3\big]^\ell\Big(\partial_h^\bullet v_h\varphi_h + v_h\partial_h^\bullet\varphi_h\Big)\,\mathrm{d}\sigma + b_h(t;V_h,\phi_h),
\end{aligned}
$$

second, using (6.26),

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} a_h(t;V_h,\phi_h)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \big[\mathcal{Q}_h^1\big]^\ell\nabla_\Gamma v_h\cdot\nabla_\Gamma\varphi_h + \big[\mathcal{Q}_h^2\big]^\ell v_h\cdot\nabla_\Gamma\varphi_h + \big[\mathcal{Q}_h^3\big]^\ell v_h\varphi_h\,\mathrm{d}\sigma\\
&= \int_{\Gamma(t)} \big[\mathcal{Q}_h^1\big]^\ell\Big(\nabla_\Gamma\partial_h^\bullet v_h\cdot\nabla_\Gamma\varphi_h + \nabla_\Gamma v_h\cdot\nabla_\Gamma\partial_h^\bullet\varphi_h\Big)\\
&\qquad + \big[\mathcal{Q}_h^2\big]^\ell\Big(\partial_h^\bullet v_h\cdot\nabla_\Gamma\varphi_h + v_h\cdot\nabla_\Gamma\partial_h^\bullet\varphi_h\Big)\\
&\qquad + \big[\mathcal{Q}_h^3\big]^\ell\Big(\partial_h^\bullet v_h\varphi_h + v_h\partial_h^\bullet\varphi_h\Big)\,\mathrm{d}\sigma\\
&\quad + \int_{\Gamma(t)} \big(\mathcal{B}(w_h,(\mathcal{Q}_h^1)^\ell - P\mathscr{A}P)\big)\nabla_\Gamma v_h\cdot\nabla_\Gamma\varphi_h\\
&\qquad + \big(\mathcal{B}_{\mathrm{adv}}(w_h,(\mathcal{Q}_h^2)^\ell - P\mathscr{B}P)\big)v_h\cdot\nabla_\Gamma\varphi_h\\
&\qquad + \big[\partial_h^\bullet[(\mathcal{Q}_h^3)^\ell - \mathscr{C}] + [(\mathcal{Q}_h^3)^\ell - \mathscr{C}]\nabla_\Gamma\cdot w_h\big]v_h\varphi_h\,\mathrm{d}\sigma\\
&\quad + \widetilde{b}_h(t;v_h,\varphi_h).
\end{aligned}
$$

Hence, we have that

$$
\begin{aligned}
&\Big|\widetilde{b}_h(t;v_h,\varphi_h) - b_h(t;V_h,\phi_h)\Big|\\
&\le c\Big(\Big\|\mathcal{B}(w_h,[\mathcal{Q}_h^1]^\ell - P\mathscr{A}P)\Big\|_{L^\infty(\Gamma_h(t))} + \Big\|\mathcal{B}_{\mathrm{adv}}(w_h,[\mathcal{Q}_h^2]^\ell - P\mathscr{B}P)\Big\|_{L^\infty(\Gamma_h(t))}\\
&\qquad + \Big\|\partial_h^\bullet([\mathcal{Q}_h^3]^\ell - \mathscr{C}) + ([\mathcal{Q}_h^3]^\ell - \mathscr{C})\nabla_\Gamma\cdot w_h\Big\|_{L^\infty(\Gamma_h(t))}\Big)\,\|v_h\|_{H^1(\Gamma(t))}\,\|\varphi\|_{H^1(\Gamma(t))}.
\end{aligned}
$$

Applying Lemma 6.15 and 6.16 in a similar fashion to the previous parts of this proof completes the result. □

Lemma 6.18.

$$\|\partial^\bullet \eta - \partial_h^\bullet \eta\|_{L^2(\Gamma(t))} \le c h^{k+1} \|\eta\|_{H^1(\Gamma(t))} \qquad \text{for } \eta \in H^1(\Gamma(t)) \tag{6.43}$$

$$\|\nabla_\Gamma(\partial^\bullet \eta - \partial_h^\bullet \eta)\|_{L^2(\Gamma(t))} \le c h^{k} \|\eta\|_{H^2(\Gamma(t))} \qquad \text{for } \eta \in H^2(\Gamma(t)). \tag{6.44}$$

Proof. We note that $w - w_h$ is a tangent vector. Then

$$\partial^\bullet \eta - \partial_h^\bullet \eta = (w - w_h) \cdot \nabla \widetilde{\eta} = (w - w_h) \cdot \nabla_\Gamma \eta.$$

We combine this calculation with (6.22) to see (6.43).

We may apply the tangential gradient to the above equation and use (6.22) again to obtain

$$\|\nabla_\Gamma(\partial^\bullet \eta - \partial_h^\bullet \eta)\|_{L^2(\Gamma(t))} \le c h^{k} \|\eta\|_{H^1(\Gamma(t))} + c h^{k+1} \|\eta\|_{H^2(\Gamma(t))}. \qquad \square$$

Proof of Theorem 6.14. We simply check the assumptions of Theorem 3.8.

We know the lift is stable from Lemma (6.10). The existence and boundedness of $\widetilde{g}_h$ and $\widetilde{b}_h^\kappa$ are dealt with in Lemma 6.13. The interpolation properties (I1) and (I2) are shown in Lemma 6.11. The geometric perturbation estimates (P1)–(P8) are shown in the sequence of Lemmas 6.17, 6.12 and 6.18. $\square$

7. Application II: A evolving open domain problem

The problem of solving parabolic problems in evolving domains has been studied for many years. In particular, we mention the arbitrary Lagrangian-Eulerian approach first proposed by Hirt, Amsden, and Cook (1974) in the context of finite difference methods and by Donea, Giuliani, and Halleux (1982); Hughes, Liu, and Zimmermann (1981) for finite element methods. Analysis of a similar problem considering both spatial and temporal discretisation is given by (Badia and Codina, 2006; Boffi and Gastaldi, 2004; Formaggia and Nobile, 1999, 2004; Gastaldi, 2001; Nobile, 2001; Gawlik and Lew, 2015). The recent analysis by Bonito, Kyza, and Nochetto (2013b,a) provides optimal order convergence for a discrete Galerkin in time approach.

We let $k \in \mathbb{N}$ be the polynomial degree of basis function we use in our finite element method that will be fixed throughout this section.

7.1. Continuous problem.

We take our notation from Section 4.1. For $t \in [0,T]$, let $\Omega(t)$ be a smoothly evolving domain and denote its boundary by $\Gamma(t) = \partial\Omega(t)$. We assume that $\Omega(t) \times \{0\}$ is an $n+1$-dimensional flat open hypersurface in $\mathbb{R}^{n+2}$ and $\Gamma(t)$ is an n-dimensional, orientable C^{k+1} hypersurface in $\mathbb{R}^{n+1}$. We will write $G \colon \bar{\Omega}_0 \times [0,T] \to \mathbb{R}^{n+1}$ for a parametrisation of $\Omega(\cdot)$ and $w(G(\cdot,t),t) = \frac{\mathrm{d}}{\mathrm{d}t} G(\cdot,t)$. We denote by Ω_T the space-time domain given by

$$\Omega_T := \bigcup_{t \in [0,T]} \Omega(t) \times \{t\}.$$

We introduce the Hilbert spaces $\mathcal{H}(t) = L^2(\Omega(t))$ and $\mathcal{V}(t) = H^1(\Omega(t))$. These spaces form a compatible pair with the linear family of homeomorphisms $\{\phi_t\}_{t\in[0,T]}$ (4.16) and $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))_{t\in[0,T]}$ form a Hilbert triple. We will also use $\mathcal{Z}_0(t) = H^2(\Omega(t))$ and $\mathcal{Z}(t) = H^{k+1}(\Omega(t))$.

We consider the following formulation of (1.5):

Problem 7.1. *Given $u_0 \in \mathcal{V}_0$, find $u \in L^2_{\mathcal{V}}$ with $\partial^\bullet u \in L^2_{\mathcal{H}}$, such that for almost every time $t \in (0,T)$ we have*

$$m(t;\partial^\bullet u,\varphi) + g(t;u,\varphi) + a(t;u,\varphi) = 0 \qquad \text{for all } \varphi \in \mathcal{V}(t)$$
$$u(\cdot,0) = u_0, \tag{7.1}$$

where for $\eta,\varphi \in \mathcal{V}(t)$, we define

$$m(t;\eta,\varphi) = \int_{\Omega(t)} \eta\varphi \,\mathrm{d}x$$
$$g(t;\eta,\varphi) = \int_{\Omega(t)} \eta\varphi\nabla\cdot w \,\mathrm{d}x$$
$$a(t;\eta,\varphi) = \int_{\Omega(t)} \mathscr{A}\nabla\eta\cdot\nabla\varphi + \mathscr{B}\eta\cdot\nabla\varphi + \mathscr{C}\eta\varphi \,\mathrm{d}x.$$

For the bilinear forms in Problem 7.1, we can apply (4.10), (4.11) and (4.13) in the case of a flat hypersurface to see that we have the transport laws

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t;\eta,\varphi) = m(t;\partial^\bullet\eta,\varphi) + m(t;\eta,\partial^\bullet\varphi) + g(t;\eta,\varphi) \qquad \text{for all } \eta,\varphi \in C^1_{\mathcal{H}} \tag{7.2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} a(t;\eta,\varphi) = a(t;\partial^\bullet\eta,\varphi) + a(t;\eta,\partial^\bullet\varphi) + b(t;\eta,\varphi) \qquad \text{for all } \eta,\varphi \in C^1_{\mathcal{V}}, \tag{7.3}$$

with the new forms

$$g(t;\eta,\varphi) = \int_{\Omega(t)} \eta\varphi\nabla\cdot w\,\mathrm{d}x \qquad \text{for } \eta,\varphi \in \mathcal{H}(t)$$

and

$$b(t;\eta,\varphi) = \int_{\Omega(t)} \mathcal{B}(w,\mathscr{A})\nabla\eta\cdot\nabla\varphi + \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B})\eta\cdot\nabla\varphi + (\partial^\bullet\mathscr{C} + \mathscr{C}\nabla\cdot w)\eta\varphi\,\mathrm{d}x \qquad \text{for } \eta,\varphi \in \mathcal{V}(t),$$

where

$$\mathcal{B}(w,\mathscr{A}) = \partial^\bullet\mathscr{A} + \nabla\cdot w\mathscr{A} + D(w,\mathscr{A})$$
$$\mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}) = \partial^\bullet\mathscr{B} + \nabla\cdot w\mathscr{B} + \sum_{j=1}^{n+1}\mathscr{B}_j\partial_{x_j} w.$$

Furthermore, it is clear that from our assumptions on w that g and b are uniformly bounded. There exists a constant $c > 0$ such that

$$|g(t;\eta,\varphi)| \le c\,\|\eta\|_{\mathcal{H}(t)}\,\|\varphi\|_{\mathcal{H}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{H}(t) \tag{7.4}$$
$$|b(t;\eta,\varphi)| \le c\,\|\eta\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{V}(t). \tag{7.5}$$

Theorem 7.2. *There exists a unique solution $u \in L^2_{\mathcal{V}}$, with $\partial^\bullet u \in L^2_{\mathcal{H}}$, to Problem 7.1 which satisfies the stability bound:*

$$\int_0^T \|u\|^2_{\mathcal{V}(t)} + \|\partial^\bullet u\|^2_{\mathcal{V}(t)}\,\mathrm{d}t \le c\,\|u_0\|^2_{\mathcal{V}_0}. \tag{7.6}$$

Proof. We simply apply the abstract theory of Theorem 2.9. It is left to show that the corresponding assumptions hold.

It is clear that (M1) and (M2) hold since $m(t;\cdot,\cdot)$ is equal to the $\mathcal{H}(t)$-inner product. The assumptions (G1) and (G2) are shown in (7.2) and (7.4). We know that the map $t \mapsto a(t;\cdot,\cdot)$ is differentiable hence measurable which shows (A1). The coercivity (A2) and boundedness (A3) of a follow from standard arguments. The existence of the bilinear form b (B1) has been shown in (7.3) and the estimate (B2) is shown in (7.5). □

7.2. **Finite element method.** The first stage of our finite element method is to define an approximate computation domain $\{\Omega(t)\}$. We follow a similar construction to the surface case presented the previous section. Our construction satisfies that the boundary Lagrange points of $\Omega_h(t)$ lie on the boundary of $\Omega(t)$ and all Lagrange points evolve with the prescribed velocity w. We will consider $\Omega_h(t)$ as an interpolant of $\Omega(t)$.

Let $\widetilde{\Omega}_{h,0}$ be a polyhedral approximation of Ω_0 equipped with a quasi-uniform, conforming subdivision $\widetilde{\mathscr{T}}_{h,0}$ (see Section 5.1 for details). We denote by $\widetilde{\Gamma}_{h,0} = \partial\widetilde{\Omega}_{h,0}$. We restrict that the vertices of $\widetilde{\Gamma}_{h,0}$ lie on the surface Γ_0. We assume that the normal projection operator (4.2), $p(\cdot,0)$ is a homomorphism from $\widetilde{\Gamma}_{h,0}$ onto Γ_0.

We extend p to construct a bijection $\Psi_h \colon \widetilde{\Omega}_{h,0} \to \Omega_0$ which we will define element-wise. We first decompose $\widetilde{\mathscr{T}}_{h,0}$ into boundary elements, which have more than one vertex on the boundary, and interior elements. For an interior element $\widetilde{K} \in \widetilde{\mathscr{T}}_{h,0}$, we define

$$\Psi_h(\widetilde{x}) = \widetilde{x} \qquad \text{for } \widetilde{x} \in \widetilde{K}.$$

Otherwise, let $\widetilde{K}$ be a boundary element and consider $\widetilde{x} \in \widetilde{K}$. Denote by $\{\widetilde{a}_i\}_{i=1}^{n+2}$ the vertices of $\widetilde{K}$ ordered so that $\{\widetilde{a}_i\}_{i=1}^{L}$ lie on Γ_0 (recall that $\Omega(t) \subset \mathbb{R}^{n+1}$). First, decompose $\widetilde{x}$ into barycentric coordinates:

$$\widetilde{x} = \sum_{j=1}^{n+2} \lambda_j(\widetilde{x})\widetilde{a}_i.$$

We introduce the function $\lambda^*(\widetilde{x})$ and the singular set σ by

$$\lambda^*(\widetilde{x}) = \sum_{j=1}^{L} \lambda_j(\widetilde{x}), \qquad \sigma = \{\widetilde{x} \in \widetilde{K} : \lambda^*(\widetilde{x}) = 0\}.$$

If $\widetilde{x} \notin \sigma$, we denote the projection onto $\widetilde{\Gamma}_{h,0} \cap \widetilde{K}$ by $y(\widetilde{x})$ given by

$$y(\widetilde{x}) = \sum_{j=1}^{L} \frac{\lambda_j(\widetilde{x})}{\lambda^*(\widetilde{x})}\widetilde{a}_i.$$

Then, we define $\Psi_h|_{\widetilde{K}}$ by

$$\Psi_h|_{\widetilde{K}}(\widetilde{x}) = \begin{cases} \widetilde{x} + (\lambda^*(\widetilde{x}))^{k+2}(p(y(\widetilde{x}),0) - y(\widetilde{x})) & \text{if } \widetilde{x} \notin \sigma \\ \widetilde{x} & \text{otherwise.} \end{cases}$$

We equip each $\widetilde{K} \in \widetilde{\mathscr{T}}_{h,0}$ with a Lagrangian standard finite element $(\widetilde{K}, \widetilde{P}, \widetilde{\Sigma})$ of order k with $\widetilde{\Sigma}$ given by evaluation at the points $\{\widetilde{a}_i\}_{i=1}^{N_K} \subset \widetilde{K}$. We write $\widetilde{I}$ for interpolation over $(\widetilde{K}, \widetilde{P}, \widetilde{\Sigma})$ and lifted the finite element onto a flat surface finite element (K, P^K, Σ^K) given by

$$\begin{aligned} K &:= \{\widetilde{I}\Psi_h(\widetilde{x}) : \widetilde{x} \in \widetilde{K}\} \\ P^K &:= \{x \mapsto \widetilde{\chi}(\widetilde{x}) : \widetilde{I}\Psi_h(\widetilde{x}), \widetilde{\chi} \in \widetilde{P}\} \\ \Sigma^K &:= \{\chi \mapsto \chi(\widetilde{I}\Psi_h(\widetilde{a}_i)), 1 \le i \le N_K\}. \end{aligned}$$

We call the union of all elements construction in this way $\mathscr{T}_{h,0}$ and call the union of element domains $\Omega_{h,0}$. Finally, we call $\{a_i\}_{i=1}^{N}$ the Lagrange nodes of $\Omega_{h,0}$.

To complete the construction, for $K_0 \in \mathscr{T}_{h,0}$, we consider the discrete domain $K(t)$ given by the discrete flow $\Phi_t^K \colon K_0 \to K(t)$ defined by

$$\Phi_t^K|_{K_0} = I_{K_0}[\Phi_t(\Psi_h(\cdot))],$$

which is a bijection onto its image. We denote its inverse by Φ_{-t}^K.

Given a flat surface finite element (K_0, P_0, Σ_0), we define a flat surface finite element $(K(t), P^K(t), \Sigma^K(t))$ at each time $t \in [0,T]$ by

$$\begin{aligned} K(t) &= \Phi_t^K(K_0) \\ P^K(t) &= \{\chi(\Phi_{-t}^K) : \chi \in P_0\} \\ \Sigma^K(t) &= \{\chi \mapsto \chi(a_i(t)) : a_i(t) = \Phi_t^K(a_i), 1 \le i \le N_K\}. \end{aligned}$$

We call the union of such elements at each time $\mathscr{T}_h(t)$ and we define $\Omega_h(t)$ by

$$\Omega_h(t) := \bigcup_{K(t) \in \mathscr{T}_h(t)} K(t),$$

and write a global discrete flow map $\Phi_t^h \colon \Gamma_{h,0} \to \Gamma_h(t)$ defined element-wise by $\Phi_t^h|_{K_0} := \Phi_t^K$. We define a global finite element space by

$$\mathcal{V}_h(t) := \{\phi_h \in C^0(\Gamma_h(t)) : \phi_h|_{K(t)} \in P^K(t) \text{ for all } K(t) \in \mathscr{T}_h(t)\}.$$

We assume that the resulting subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ is a uniformly quasi-uniform. In order to show the result of this construction satisfies the other assumptions we require we first state a result shown by Elliott and Ranner (2013):

Lemma 7.3. *The mapping $\Psi_K = \Psi_h|_{\widetilde{K}}$ is of class C^2 when restricted to each element $\widetilde{K} \in \widetilde{\mathscr{T}}_{h,0}$ and satisfies*

$$\|D^m \Psi_K\|_{L^\infty(\widetilde{K})} \le ch^{2-m} \qquad \text{for } 0 \le m \le 2. \tag{7.7}$$

Proof. We combine the result of Elliott and Ranner (2013, Proposition 4.4) show for $\Psi_K(F_{\widetilde{K}}(\cdot)) \colon \hat{K} \to \mathbb{R}^{n+1}$ with the rescaling result (5.7) to see the desired result. □

We will assume that this construction results in a uniformly quasi-uniform evolving subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ - that is that the velocity w is such that the mesh does not become too distorted.

Proposition 7.4. *The subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ forms a uniformly quasi-uniform, evolving conforming subdivision of $\{\Gamma_h(t)\}_{t\in[0,T]}$ and $\{\mathcal{V}_h(t)\}_{t\in[0,T]}$ is an evolving surface finite element space consisting of k-surface finite elements.*

Proof. The proof follows in the same way as Proposition 6.4. The only part to check is that the discrete flow map Φ_t^h is uniformly bounded in $W^{2,\infty}(\mathscr{T}_{h,0})$. This follows from the definition of Φ_t^h and the smoothness of Ψ_K (7.7) and and the smooth flow map Ψ_t. □

The element flow map Φ_t^K defines a velocity on each element W_K by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^K(\cdot) = W_K(\Phi_t^K(\cdot), t) \qquad \text{for } t \in [0,T].$$

This can be combined into a global velocity W_h. We note that the global velocity is determined purely by the velocity of the vertices $\{a_i(t)\}_{i=1}^N$:

$$W_h(x,t) = \sum_{i=1}^N w(a_i(t),t)\chi_i(x,t) \qquad \text{for } x \in \Omega_h(t). \tag{7.8}$$

The push-forward map defines, in an element-wise fashion, a strong material derivative on $\{\Omega_h(t)\}_{t \in [0,T]}$ which we can write as

$$\partial_h^\bullet V_h = \partial_t \widetilde{V}_h + W_h \cdot \nabla \widetilde{V}_h \quad \text{for } V_h \in C^1_{\mathcal{V}_h}$$

with the usual convention that $\widetilde{V}_h$ is the smooth extension of V_h to a neighbourhood of $\Omega_h(t)$.

The finite element method is based on the variational formulation (2.6) of Problem 7.1. We introduce a element-wise smooth $(n+1) \times (n+1)$-diffusion tensor $\mathscr{A}_h$, an element-wise smooth $(n+1)$ dimensional vector field $\mathscr{B}_h$ and an element-wise smooth scalar field $\mathscr{C}_h$. We will use the notation $\mathscr{A}_K := \mathscr{A}_h|_{K(t)}, \mathscr{B}_K := \mathscr{B}_h|_{K(t)}$ and $\mathscr{C}_K := \mathscr{C}_h|_{K(t)}$, for all $K(t) \in \mathscr{T}_h(t)$, which we assume satisfy:

$$\sup_{h \in (0,h_0)} \sup_{t \in [0,T]} \left(\|\mathscr{A}_h\|_{L^\infty(\Omega_h(t))} + \|\mathscr{B}_h\|_{L^\infty(\Omega_h(t))} + \|\mathscr{C}_h\|_{L^\infty(\Omega_h(t))} \right) \leq C.$$

Problem 7.5. *Given $U_{h,0} \in \mathcal{V}_{h,0}$, find $U_h \in C^1_{\mathcal{V}_h}$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t; U_h, \phi_h) + a_h(t; U_h, \phi_h) = m_h(t; U_h, \partial_h^\bullet \phi_h) \quad \textit{for all } \phi_h \in C^1_{\mathcal{V}_h} \tag{7.9}$$
$$U_h(\cdot, 0) = U_{h,0},$$

where for $V_h, \phi_h \in \mathcal{V}_h(t)$ we have

$$m_h(t; V_h, \phi_h) = \int_{\Omega_h(t)} V_h \phi_h \,\mathrm{d}x$$
$$a_h(t; V_h, \phi_h) = \sum_{K(t) \in \mathscr{T}_h(t)} \int_{K(t)} \mathscr{A}_h \nabla V_h \cdot \nabla \phi_h + \mathscr{B}_h V_h \cdot \nabla \phi_h + \mathscr{C}_h V_h \phi_h \,\mathrm{d}x.$$

7.3. **Stability.** To understand this problem, we first introduce some discrete norms on $\mathcal{V}_h(t)$:

$$\|\chi_h\|_{\mathcal{V}_h(t)} := \|\chi_h\|_{H^1(\mathscr{T}_h(t))} = \left(\sum_{K(t) \in \mathscr{T}_h(t)} \int_{K(t)} |\nabla \chi_h|^2 + \chi_h^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$
$$\|\chi_h\|_{\mathcal{H}_h(t)} := \|\chi_h\|_{L^2(\Omega_h(t))} = \left(\int_{\Omega_h(t)} \chi_h^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

We note that the assumption that $\{\mathscr{T}_h(t)\}_{t \in [0,T]}$ is uniformly quasi-uniform implies that $\{\mathcal{V}_h(t), \phi_h^t\}_{t \in [0,T]}$ is compatible when equipped with the $\mathcal{V}_h(t)$ or $\mathcal{H}(t)$-norms (Lemma 5.22) for ϕ_h^t the push-forward map defined by Φ_t^h:

$$\phi_h^t(\chi_h)(x) = \chi_h(\Phi_{-t}^h)(x) \quad \text{for } x \in \Gamma_h(t), \chi_h \in \mathcal{V}_{h,0}.$$

Lemma 7.6. *The discrete velocity W_h of the discrete evolving domain $\{\Omega_h(t)\}$ is uniformly bounded in $W^{1,\infty}(\Omega_h(t))$. That is, there exists a constant $C > 0$ such that for all $h \in (0, h_0)$*

$$\sup_{t \in [0,T]} \|W_h\|_{W^{1,\infty}(\mathscr{T}_h(t))} \leq C. \tag{7.10}$$

Proof. The bound follows using the characterisation (7.8) by using the interpolation bound shown in Corollary 5.27. □

We have a transport formula for the domain $\{\Omega_h(t)\}$.

Lemma 7.7. *There exists bilinear forms $g_h(t;\cdot,\cdot), b_h(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t)\to\mathbb{R}$ such that for all $Z_h,\chi_h\in C^1_{\mathcal{V}_h}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(t;Z_h,\chi_h) = m_h(t;\partial^\bullet_h Z_h,\chi_h) + m_h(t;Z_h;\partial^\bullet_h \chi_h) + g_h(t;Z_h,\chi_h) \tag{7.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_h(t;Z_h,\chi_h) = a_h(t;\partial^\bullet_h Z_h,\chi_h) + a_h(t;Z_h;\partial^\bullet_h \chi_h) + b_h(t;Z_h,\chi_h), \tag{7.11b}$$

where

$$g_h(t;Z_h,\chi_h) := \int_{\Omega_h(t)} Z_h\phi_h \nabla\cdot W_h \,\mathrm{d}x,$$

and

$$\begin{aligned} b_h(t;Z_h,\chi_h) := \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} & \mathcal{B}_h(W_h,\mathscr{A}_h)\nabla Z_h\cdot\nabla\chi_h + \mathcal{B}_{\mathrm{adv},h}(W_h,\mathscr{A}_h) Z_h\cdot\nabla\chi_h \\ & + (\partial^\bullet_h \mathscr{C}_h + \mathscr{C}_h\nabla\cdot W_h) Z_h\chi_h \,\mathrm{d}x. \end{aligned}$$

Furthermore, there exists a constants $c>0$ such that for all $t\in[0,T]$ and all $h\in(0,h_0)$ we have for all $Z_h,\chi_h\in\mathcal{V}_h(t)$

$$\begin{aligned} |g_h(t;Z_h,\chi_h)| &\le c\,\|Z_h\|_{\mathcal{H}_h(t)}\,\|\chi_h\|_{\mathcal{H}_h(t)} \\ |b_h(t;Z_h,\chi_h)| &\le c\,\|Z_h\|_{\mathcal{V}_h(t)}\,\|\chi_h\|_{\mathcal{V}_h(t)}\,. \end{aligned}$$

Proof. The bilinear forms exist due to the more general Lemma 5.31 applied to a flat domain. The estimates follow from the previous Lemma. □

Theorem 7.8. *There exists a unique solution of the finite element scheme (7.9). The solution U_h satisfies the stability bound:*

$$\sup_{t\in(0,T)} \|U_h\|^2_{\mathcal{H}_h(t)} + \int_0^T \|U_h\|^2_{\mathcal{V}_h(t)}\,\mathrm{d}t \le c\,\|U_{h,0}\|^2_{\mathcal{H}_h(t)}\,. \tag{7.12}$$

Proof. We apply the abstract result of Theorem 3.3. It is left to check the required assumptions.

The assumptions on m_h, (M_h1) and (M_h2), follow directly since m_h is equal to the $\mathcal{H}_h(t)$ inner-product. The estimates on a_h, (A_h2) and (A_h3) follow in the same manner as Theorem 7.2. The transport formulae and estimates for g_h and b_h, (G_h1), (G_h2) (B_h1) and (B_h2), are shown in Lemma 7.7. □

7.4. **Error analysis.** We construct a bijection between the computation domain $\Omega_h(t)$ and the continuous problem domain $\Omega(t)$ which we will call the lifting operator. We do this using a similar construction to Ψ_h used to define $\Omega_h(t)$ at the start of Section 7.2. It will again be based on using an extension of the normal projection operator used as a lifting operator in Section 6.4.

Fix $t\in[0,T]$. We wish to construct a bijection $\Lambda_h(\cdot,t)\colon\Omega_h(t)\to\Omega(t)$ which we will define element-wise. We decompose $\mathscr{T}_h(t)$ into boundary elements, which have more than one vertex on the boundary, and interior elements. For an interior element $K(t)\in\mathscr{T}_h(t)$, we define

$$\Lambda_h(x,t) = x \qquad \text{for } x\in K(t).$$

Otherwise, let $K(t)$ be a boundary element and consider $x\in K(t)$. Denote by $\{a_i(t)\}_{i=1}^{n+2}$ the vertices of $K(t)$ ordered so that $\{a_i(t)\}_{i=1}^{L}$ lie on $\Gamma(t)$. We recall that the element $K(t)$

is given by a parametrisation F_K over a reference element $\hat{K}$ so that we can define points $\hat{x}$ and vertices $\{\hat{a}_i\}_{i=1}^{n+2}$ in $\hat{K}$ by

$$x = F_K(\hat{x}, t) \quad \text{and} \quad a_i(t) = F_K(\hat{a}_i, t) \text{ for } 1 \leq i \leq n+2.$$

We decompose $\hat{x}$ into barycentric coordinates on $\hat{K}$:

$$\hat{x} = \sum_{i=1}^{n+2} \hat{\lambda}_i(\hat{x}) \hat{a}_i.$$

We introduce the function $\hat{\lambda}^*(\hat{x})$ and the singular set $\hat{\sigma}$ by

$$\hat{\lambda}^*(\hat{x}) = \sum_{i=1}^{L} \hat{\lambda}_j(\hat{x}), \qquad \hat{\sigma} = \{\hat{x} \in \hat{K} : \hat{\lambda}^*(\hat{x}) = 0\}.$$

If $x \notin F_K(\hat{\sigma}, t)$, we denote the projection onto $\Gamma_h(t) \cap K(t)$ by $y(x)$ given by

$$y(x) = F_K(\hat{y}(\hat{x}), t), \qquad \hat{y}(\hat{x}) = \sum_{i=1}^{L} \frac{\hat{\lambda}_j(\hat{x})}{\hat{\lambda}^*(\hat{x})} \hat{a}_i.$$

We then define $\Lambda_h(\cdot, t)|_{K(t)}$ by

$$\Lambda_h(\cdot, t)|_{K(t)}(x) = \begin{cases} x + (\lambda^*(\hat{x}))^{k+2} (p(y(x), t) - y(x)) & \text{if } x \notin F_K(\hat{\sigma}, t) \\ x & \text{otherwise.} \end{cases}$$

We next follow a sequence of calculations to show the properties of Λ_h. These estimates are based on previous work by Bernardi (1989) and Elliott and Ranner (2013). It is useful to recall the following formula (Bernardi, 1989, Eq. 2.9) for two smooth functions f, g

$$D^m(f \circ g) = \sum_{r=1}^{m} D^r f \left(\sum_{\underline{i} \in E(m,r)} c_{\underline{i}} \prod_{q=1}^{m} (D^q g)^{i_q} \right), \tag{7.13}$$

where $E(m,r)$ is the set given by

$$E(m,r) = \left\{ \underline{i} \in \mathbb{N}^m : \sum_{q=1}^{m} i_q = r \text{ and } \sum_{q=1}^{m} q i_q = m \right\}.$$

A direct calculation shows that

$$\|D_{\hat{x}}^m \hat{y}\|_{L^\infty(\hat{K} \setminus \hat{\sigma})} \leq \frac{c}{(\lambda^*(\hat{x}))^m},$$

for a constant c independent of $\hat{x}$ and $K(t)$. Then, we have that

$$\begin{aligned} \|D_{\hat{x}}^m y\|_{L^\infty(\hat{K} \setminus \hat{\sigma})} &\leq c \sum_{r=1}^{m} \|D_{\hat{x}}^r F_K(\cdot, t)\|_{L^\infty(\hat{K})} \left(\sum_{\underline{i} \in E(m,r)} \prod_{q=1}^{m} \|D_{\hat{x}}^q \hat{y}\|_{L^\infty(\hat{K} \setminus \sigma)}^{i_q} \right) \\ &\leq \sum_{r=1}^{m} \frac{h_K^r}{(\lambda^*(\hat{x}))^m} \leq c \frac{h_K}{(\lambda^*(\hat{x}))^m}, \end{aligned} \tag{7.14}$$

where in the second line, we have used that (see Lemma 5.11)

$$\|D_{\hat{x}}^r F_K(\cdot, t)\|_{L^\infty(\hat{K})} \leq c \|A_K\|^r \leq c h_K^r.$$

Next, applying (7.13) and (7.14), we have

$$\left\| D_{\hat{x}}^m \big(p(y(\cdot), t) - y(\cdot) \big) \right\|_{L^\infty(\hat{K} \setminus \hat{\sigma})} \leq c \sum_{r=1}^{m} \left\| D_y^r \big(p(\cdot, t) - \mathrm{Id} \big) \right\|_{L^\infty(K(t))} \frac{h_K^r}{(\lambda^*(\hat{x}))^m}.$$

Using a similar geometric construction to Lemma 6.15, we have

$$\left\| D^r\big(p(\cdot,t) - \mathrm{Id}\big)\right\|_{L^\infty(K(t))} = \|D^r d\|_{L^\infty(K(t))} \le c h_K^{k+1-r}.$$

Hence, we infer that

$$\left\| D^m_{\hat{x}}\big(p(y(\cdot),t) - y(\cdot)\big)\right\|_{L^\infty(\hat{K}\setminus\hat{\sigma})} \le c\frac{h_K^{k+1}}{(\lambda^*(\hat{x}))^m}.$$

Finally, using the Leibniz formula, we have

$$\begin{aligned}\left| D^m_{\hat{x}}\big((\lambda^*(\hat{x}))^m(p(y(\hat{x}),t) - y(\hat{x}))\big)\right| &\le c\sum_{r=0}^{m}(\lambda^*(\hat{x}))^{k+2-r}\left|D^{m-r}_{\hat{x}}\big(p(y(\cdot),t)-y(\cdot)\big)\right| \\ &\le c\sum_{r=0}^{m}(\lambda^*(\hat{x}))^{k+2-r}\frac{h_K^{k+1}}{(\lambda^*(\hat{x}))^{m-r}} \\ &\le c h_K^{k+1}(\lambda^*(\hat{x}))^{k+2-m}.\end{aligned}$$

Lemma 7.9. *The lifting function $\Lambda_h(\cdot,t)\colon \Omega_h(t) \to \Omega(t)$ is an element-wise C^{k+1}-diffeomorphism and satisfies*

$$\sup_{h\in(0,h_0)}\sup_{t\in[0,T]}\|\Lambda_h(\cdot,t)\|_{W^{k+1,\infty}(\Omega_h(t))} \le C. \tag{7.15}$$

Furthermore, there exists constants $c_1, c_2 > 0$ such that for all $h \in (0,h_0)$ and all $t \in [0,T]$

$$c_1 \le \inf_{x\in\Omega_h(t)}\|D\Lambda_h(x,t)\| \le \sup_{x\in\Omega_h(t)}\|D\Lambda_h(x,t)\| \le c_2. \tag{7.16}$$

Proof. The smoothness follows from the fact that Λ_h restricted to each element is the composition of smooth functions. The result is clear for all internal elements. Consider a time $t \in [0,T]$ and a single boundary element $K(t) \in \mathscr{T}_h(t)$. The above calculations, combined with (5.7) and (5.5b), show that

$$\begin{aligned}\left\| D^m\Lambda_h|_{K(t)}(\cdot,t) - \mathrm{Id}\right\|_{L^\infty(K(t))} &\le \frac{c}{\rho_K^m}\left\|D^m\Lambda_h|_{K(t)}(F_K(\cdot,t),t) - \mathrm{Id}\right\|_{L^\infty(\hat{K})} \\ &\le c\frac{h_K^{k+1}}{\rho_K^m}(\lambda^*(\hat{x}))^m \le c\frac{h_K^{k+1}}{\rho_K^m}.\end{aligned}$$

Since, we have assumed that $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ forms a uniformly regular subdivision, we have $\rho_K < \rho h_K$. We complete the proof by taking h sufficiently small and using the inverse function theorem. □

We will also require bounds on the time derivative of Λ_h. We consider an element $K(t) \in \mathscr{T}_h(t)$ and the trajectory of a point $X(t)$ which follows the velocity field W_h. From the definition of W_h, we have that

$$X(t) = F_K(\hat{x},t), \qquad \hat{x} = \sum_{j=1}^{n+2}\lambda_j\hat{a}_i.$$

In particular, the barycentric coordinate representation of $X(t)$ do not depend on time. Therefore, writing $x = X(t)$ we have

$$\partial_h^\bullet y(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}y(X(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}F_K(\hat{y}(\hat{x}),t) = \frac{\partial F_K}{\partial t}(\hat{y}(\hat{x}),t) = W_h(y(x,t),t).$$

Then we can compute that if $K(t)$ is a boundary element we have

$$
\begin{aligned}
\partial_h^\bullet \Lambda_h|_{K(t)}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t} \Lambda_h|_{K(t)}(X(t),t) \\
&= \begin{cases} W_h(x,t) + (\lambda^*(x))^{k+2} \left(\frac{\partial p}{\partial t}(y,t) + \nabla p(y,t) W_h(y(x,t),t) - W_h(y(x,t),t) \right) & \text{if } x \notin F_K(\hat\sigma,t) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} W_h(x,t) + (\lambda^*(x))^{k+2} \left(\big(W_h(y,t) - w(y,t)\big) \cdot \nu(y,t)\nu(y,t) - d(y,t)\nu_t(y,t) \right) & \text{if } x \notin F_K(\hat\sigma,t) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

A similar calculation to those prior to Lemma 7.9 show that

$$
\|\partial_h^\bullet \Lambda_h(\cdot,t) - W_h(\cdot,t)\|_{W^{m,\infty}(K(t))} \le c h_K^{k+1-m} (\lambda^*(\hat x))^{k+1-m} \le c h_K^{k+1-m}. \tag{7.17}
$$

This follows by using the smoothness of the surface $\Gamma(t)$ along with the fact that W_h is an interpolant of w (Corollary 5.27).

For $t \in [0,T]$ and a function $\chi_h \in \mathcal{V}_h(t)$, we define its lift $\chi_h^\ell \colon \Omega(t) \to \mathbb{R}$ by

$$
\chi_h^\ell(\Lambda_h(x,t)) = \chi_h(x) \qquad \text{for } x \in \Gamma_h(t).
$$

We will also make use of an inverse lift for continuous functions on $\Omega(t)$. For $\eta \in C(\Omega(t))$, we define the inverse lift of η, denoted by $\eta^{-\ell}$ by

$$
\eta^{-\ell}(x) = \eta(\Lambda_h(x,t),t) \qquad \text{for } x \in \Gamma_h(t).
$$

Lemma 7.10. *Let $\chi_h \in \mathcal{V}_h(t)$ and denote its lift by χ_h^ℓ. Then there exists constants $c_1, c_2 > 0$ such that*

$$
c_1 \left\| \chi_h^\ell \right\|_{\mathcal{H}(t)} \le \|\chi_h\|_{\mathcal{H}_h(t)} \le c_2 \left\| \chi_h^\ell \right\|_{\mathcal{H}(t)} \tag{7.18}
$$

$$
c_1 \left\| \chi_h^\ell \right\|_{\mathcal{V}(t)} \le \|\chi_h\|_{\mathcal{V}_h(t)} \le c_2 \left\| \chi_h^\ell \right\|_{\mathcal{V}(t)}. \tag{7.19}
$$

Proof. We apply Lemma 5.25 using the previous Lemma. □

We define the space of lifted functions to be $\mathcal{V}_h^\ell(t)$ given by

$$
\mathcal{V}_h^\ell(t) := \{ \chi_h^\ell : \chi_h \in \mathcal{V}_h(t) \}.
$$

We will also make use of the notation that $\mathcal{Z}_0^{-\ell}(t) = \{ \eta^{-\ell} : \eta \in \mathcal{Z}_0(t) \}$. The space $\mathcal{V}_h^\ell(t)$ is equipped with the following approximation property.

Lemma 7.11. *For $\eta \in C(\Omega(t))$ there exists a Lagrangian interpolation operator $I_h \eta \in \mathcal{V}_h(t)$ that is well defined. Furthermore, the following bounds hold for constants independent of h and time:*

$$
\|\eta - I_h\eta\|_{L^2(\Omega(t))} + h \|\nabla(\eta - I_h\eta)\|_{L^2(\Omega(t))} \le c h^{k+1} \|\eta\|_{\mathcal{Z}(t)} \qquad \textit{for } \eta \in \mathcal{Z}(t) \tag{7.20}
$$

$$
\|\eta - I_h\eta\|_{L^2(\Omega(t))} + h \|\nabla(\eta - I_h\eta)\|_{L^2(\Omega(t))} \le c h^{2} \|\eta\|_{\mathcal{Z}_0(t)} \qquad \textit{for } \eta \in \mathcal{Z}_0(t). \tag{7.21}
$$

Proof. We may simply apply the result of Theorem 5.26. □

We can also use the lift to define an evolving lifted triangulation. For each $t \in [0,T]$ and $h \in (0,h_0)$, we define

$$
\mathscr{T}_h^\ell(t) := \{ K^\ell(t) : K(t) \in \mathscr{T}_h(t) \}.
$$

The edges of these curvilinear-simplicies evolve with a velocity w_h which can be characterised as follows. Let $X(t)$ be the trajectory of a point on $\Gamma_h(t)$ according to the flow Φ^h_t. Then we have that

$$W_h(X(t),t) = \frac{\mathrm{d}}{\mathrm{d}t} X(t).$$

Now consider a point $Y(t) = \Lambda_h(X(t),t)$. The trajectory of $Y(t)$ defines the velocity field w_h by

$$w_h(Y(t),t) := \frac{\mathrm{d}}{\mathrm{d}t} Y(t) = (\partial_t \Lambda_h)(X(t),t) + (\nabla \Lambda_h)(X(t),t) W_h(X(t),t). \tag{7.22}$$

Equivalently this defines a flow Φ^ℓ_t which is the map given by

$$\Phi^\ell_{t^*}(y_0) = Y(t^*) \quad \text{such that} \quad \frac{\mathrm{d}}{\mathrm{d}t} Y(t) = w_h(Y(t),t), Y(0) = y_0.$$

In turn, this flow defines a push-forward map ϕ^ℓ_t on $\{\mathcal{H}(t)\}$ given by

$$\phi^\ell_t(\eta)(x) = \eta(\Phi^\ell_{-t}(x)) \qquad x \in \Omega(t), \eta \in \mathcal{H}_0.$$

Lemma 7.12. *The pairs $\{\mathcal{H}(t), \phi^\ell_t\}_{t\in[0,T]}$ and $\{\mathcal{V}(t), \phi^\ell_t\}_{t\in[0,T]}$ are compatible hence we may define a material derivative $\partial^\bullet_h \eta$ for $\eta \in C^1_{\mathcal{H}}$ and transport formula: There exists a bilinear for $\widetilde{g}_h(t;\cdot,\cdot) \colon \mathcal{H}(t) \times \mathcal{H}(t) \to \mathbb{R}$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t;\eta,\varphi) = m(t;\partial^\bullet_h \eta,\varphi) + m(t;\eta,\partial^\bullet_h \varphi) + \widetilde{g}_h(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{H}}, \tag{7.23}$$

and there exists a constant $c > 0$ such that for all $t \in [0,T]$ and $h \in (0,h_0)$ we have

$$|\widetilde{g}_h(t;\eta,\varphi)| \le c \|\eta\|_{\mathcal{H}(t)} \|\varphi\|_{\mathcal{H}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{H}(t). \tag{7.24}$$

Furthermore, we have a new transport formula for the a bilinear form. There exists a bilinear form $\widetilde{b}_h(t;\cdot,\cdot) \colon \mathcal{V}(t) \times \mathcal{V}(t) \to \mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t;\eta,\varphi) = a(t;\partial^\bullet_h \eta,\varphi) + a(t;\eta,\partial^\bullet_h \varphi) + \widetilde{b}_h(t;\eta,\varphi) \qquad \text{for } \eta,\varphi \in C^1_{\mathcal{V}}, \tag{7.25}$$

and there exists a constant $c > 0$ such that for all $t \in [0,T]$ and $h \in (0,h_0)$ we have

$$\left|\widetilde{b}_h(t;\eta,\varphi)\right| \le c \|\eta\|_{\mathcal{V}(t)} \|\varphi\|_{\mathcal{V}(t)} \qquad \text{for all } \eta,\varphi \in \mathcal{V}(t). \tag{7.26}$$

Proof. We simply apply Lemma 5.32. □

We next show some geometric estimates arising from the use of the lifting function Λ_h.

Lemma 7.13. *Under the above assumptions, we have the estimates that*

$$\sup_{t\in[0,T]} \|D\Lambda_h(\cdot,t) - \mathrm{Id}\|_{L^\infty(\Omega_h(t))} \le ch^k \tag{7.27}$$

$$\sup_{t\in[0,T]} \|\partial^\bullet_h D\Lambda_h(\cdot,t)\|_{L^\infty(\Omega_h(t))} \le ch^k. \tag{7.28}$$

Additionally, writing $J_h(\cdot,t) = \sqrt{\det\big((D\Lambda_h(\cdot,t))^t (D\Lambda_h(\cdot,t))\big)}$, we have

$$\sup_{t\in[0,T]} \|J_h(\cdot,t) - 1\|_{L^\infty(\Omega_h(t))} \le ch^k \tag{7.29}$$

$$\sup_{t\in[0,T]} \|\partial^\bullet_h J_h(\cdot,t)\|_{L^\infty(\Omega_h(t))} \le ch^k. \tag{7.30}$$

Proof. We have already shown (7.27) as part of the proof of Lemma 7.9. The second result then follows by a result of (Ipsen and Rehman, 2008).

To show the time derivative bounds, we have

$$\partial_h^\bullet D\Lambda_h = D\partial_h^\bullet \Lambda_h - (DW_h)(D\Lambda_h).$$

Then applying (7.17), (7.27) together with Lemma 7.6, we have

$$\begin{aligned}\|\partial_h^\bullet D\Lambda_h\|_{L^\infty(\Omega_h(t))} &\le \|D(\partial_h^\bullet \Lambda_h - W_h)\|_{L^\infty(\Omega_h(t))} + \|DW_h(\mathrm{Id} - D\Lambda_h)\|_{L^\infty(\Omega_h(t))} \\ &\le ch^k + \|DW_h\|_{L^\infty(\Omega_h(t))} ch^k \le ch^k.\end{aligned}$$

This shows (7.28).

For (7.30) we have, applying (7.27) and (7.28), that

$$\begin{aligned}&\left|\partial_h^\bullet \sqrt{\det\big((D\Lambda_h(\cdot,t))^t (D\Lambda_h(\cdot,t))\big)}\right| \\ &= \left|\sqrt{\det\big((D\Lambda_h(\cdot,t))^t (D\Lambda_h(\cdot,t))\big)} \Big(\big((D\Lambda_h(\cdot,t))^t (D\Lambda_h(\cdot,t))\big)^{-1} \operatorname{trace} \partial_h^\bullet D\Lambda_h\Big)\right| \\ &\le c\,\|\partial_h^\bullet D\Lambda_h\| \le ch^k.\end{aligned}$$

□

These bounds allow to show some of the abstract error bounds required use the result of Theorem 3.8.

Lemma 7.14. *We have the estimate:*

$$|w - w_h|_{L^\infty(\Omega(t))} + h\,|\nabla(w - w_h)|_{L^\infty(\Omega(t))} \le ch^{k+1}. \tag{7.31}$$

This implies that

$$\|\partial^\bullet \eta - \partial_h^\bullet \eta\|_{L^2(\Omega(t))} \le ch^{k+1} \qquad \textit{for } \eta \in C^1_{\mathcal{V}} \tag{7.32a}$$

$$\|\nabla(\partial^\bullet \eta - \partial_h^\bullet \eta)\|_{L^2(\Omega(t))} \le ch^k \qquad \textit{for } \eta \in C^1_{\mathcal{Z}_0}. \tag{7.32b}$$

Proof. We write for $x \in \Omega_h(t)$ that

$$\begin{aligned}&w(\Lambda_h(x,t),t) - w_h(\Lambda_h(x,t)) \\ &= \big(w(\Lambda_h(x,t),t) - W_h(x,t)\big) + \big(W_h(x,t) - w_h(x,t)\big) \\ &= \big(w(\Lambda_h(x,t),t) - W_h(x,t)\big) + \big(W_h(x,t) - \partial_h^\bullet \Lambda_h(x,t)\big).\end{aligned}$$

Hence, we can apply interpolation theorem (Corollary 5.27) and the estimate (7.17) to achieve the estimate (7.31). The bounds (7.32) follow from a simple calculation and the previous estimate. □

Lemma 7.15. *There exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $h \in (0,h_0)$ the following holds for all $V_h, \phi_h \in \mathcal{V}_h(t) + \mathcal{Z}_0^{-\ell}(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t) + \mathcal{Z}_0^{-\ell}(t)$:*

$$|m(t;v_h,\varphi_h) - m_h(t;V_h,\phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{7.33}$$

$$|\widetilde{g}_h(t;v_h,\varphi_h) - g_h(t;V_h,\phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{7.34}$$

$$|a(t;v_h,\varphi_h) - a_h(t;V_h,\phi_h)| \le ch^k \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{7.35}$$

$$\left|\widetilde{b}_h(t;v_h,\varphi_h) - b_h(t;V_h,\phi_h)\right| \le ch^k \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{7.36}$$

$$\left|\widetilde{b}_h(t;v_h,\varphi_h) - b(t;v_h,\varphi_h)\right| \le ch^k \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}. \tag{7.37}$$

For $\eta, \varphi \in \mathcal{Z}_0(t)$ with inverse lifts $\eta^{-\ell}, \varphi^{-\ell}$, we have

$$\left|a(t;\eta,\varphi) - a_h(t;\eta^{-\ell},\varphi^{-\ell})\right| \leq ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \tag{7.38}$$

$$\left|\widetilde{b}_h(t;\eta,\varphi) - b_h(t;\eta^{-\ell},\varphi^{-\ell})\right| \leq ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \tag{7.39}$$

$$\left|a(t;\partial_h^\bullet\eta,\varphi) - a_h(t;\partial_h^\bullet\eta^{-\ell},\varphi^{-\ell})\right| \leq ch^{k+1}(\|\eta\|_{\mathcal{Z}_0(t)} + \|\partial^\bullet\eta\|_{\mathcal{Z}_0(t)}) \|\varphi\|_{\mathcal{Z}_0(t)} \tag{7.40}$$

Proof. We use the notation $J_h = \sqrt{\det\left((D\Lambda_h)^t(D\Lambda_h)\right)}$. For (7.33), we have

$$\int_{\Omega(t)} v_h \varphi \,\mathrm{d}x = \int_{\Omega_h(t)} V_h \phi_h J_h \,\mathrm{d}x.$$

Hence, we have, applying (7.29) and Lemma 7.10, that

$$\begin{aligned} |m(t;v_h,\varphi_h) - m_h(t;V_h,\phi_h)| &= \left|\int_{\Omega_h(t)} V_h\phi_h\,(J_h - 1)\right| \mathrm{d}x \\ &\leq ch^k \|v_h\|_{\mathcal{H}_h(t)} \|\varphi_h\|_{\mathcal{H}_h(t)}. \end{aligned}$$

Applying the narrow band trace inequality Lemma 4.2 this can be improved to

$$|m(t;v_h,\varphi_h) - m_h(t;V_h,\phi_h)| \leq ch^{k+1} \|v_h\|_{\mathcal{V}_h(t)} \|\varphi_h\|_{\mathcal{V}_h(t)}.$$

Similarly, for (7.35), we have

$$\begin{aligned} &\int_{\Omega(t)} \mathscr{A}_h \nabla v_h \cdot \nabla\varphi_h + \mathscr{B}_h v_h \cdot \nabla\varphi_h + \mathscr{C}_h v_h \varphi_h \,\mathrm{d}x \\ &= \int_{\Omega_h(t)} J_h (D\Lambda_h)\mathscr{A}(D\Lambda_h)^t \nabla V_h \cdot \nabla\phi_h + J_h(D\Lambda_h)\mathscr{B}_h V_h \cdot \nabla\phi \\ &\qquad + J_h V_h \phi_h \,\mathrm{d}x. \end{aligned}$$

Applying (7.27), (7.29) and Lemma 7.10 we have the desired result. Again, by applying Lemma 4.2 we show the improved bound in (7.38).

We apply a similar process to the proof of Lemma 6.17 combined with the results of Lemma 7.13 and the narrow band trace inequality (Lemma 4.2) to show the estimates (7.34), (7.36) and (7.39).

Finally, (7.37) follows from the estimate (7.31). The bound (7.40) follows from (7.38), the fact that $(\partial_h^\bullet\eta)^{-\ell} = \partial_h^\bullet\eta^{-\ell}$ and the estimate (7.32). □

Finally, we have collected all the estimates we require to show the error bound.

Theorem 7.16. *Let $\mathscr{A}_h = \mathscr{A}^{-\ell}$, $\mathscr{B}_h = \mathscr{B}^{-\ell}$ and $\mathscr{C}_h = \mathscr{C}^{-\ell}$ and let $u \in L^2_{\mathcal{V}}$ be the solution of* (7.1) *which satisfies*

$$\sup_{t\in(0,T)} \|u\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet u\|^2_{\mathcal{Z}(t)} \,\mathrm{d}t \leq C_u. \tag{7.41}$$

Let $U_h \in C^1_{\mathcal{V}_h}$ be the solution of the finite element scheme (7.9) *and denote its lift by $u_h = U_h^\ell$. Then we have the following error estimate*

$$\sup_{t\in(0,T)} \|u - u_h\|^2_{\mathcal{H}(t)} + h^2 \int_0^T \|u - u_h\|^2_{\mathcal{V}(t)} \,\mathrm{d}t \leq c\left\|u - u_{h,0}\right\|^2_{\mathcal{H}(t)} + ch^{2k+2} C_u. \tag{7.42}$$

Proof. The proof is performed by applying the abstract result from Theorem 3.8.

We know the lift is stable from Lemma (7.10). The existence and boundedness of $\widetilde{g}_h$ and $\widetilde{b}_h^\kappa$ are dealt with in Lemma 7.12. The interpolation properties (I1) and (I2) are shown in

Lemma 7.11. The geometric perturbation estimates (P1)–(P8) are shown in Lemmas 7.15 and 7.14. □

8. Application III: A coupled bulk-surface problem

8.1. **Notation and setting.** In this section we will consider a coupled bulk-surface problem. See the work of Elliott and Ranner (2013); Gross, Olshanskii, and Reusken (2015); Burman et al. (2016) for the approaches to stationary surface problems. The functional analytic setting will be the product of spaces over the bulk domain, $\Omega(t)$, and the surface, $\Gamma(t)$. Before we start to tackle this problem, we introduce some further notation based on the content of Section 4.1.

For $t \in [0,T]$, let $\Omega(t)$ be a smoothly evolving domain with smoothly evolving boundary $\Gamma(t) = \partial\Omega(t)$ with evolution defined by the smooth flow $\Phi_t \colon \bar{\Omega}_0 \to \bar{\Omega}(t)$. Precisely, we consider $\{\Omega(t)\}_{t\in[0,T]}$ to be a evolving flat hypersurface which satisfies the assumptions in Section 4.2.2. In particular, we assume that there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \|w\|_{W^{k+1,\infty}(\Omega(t))} + \left\|w|_{\Gamma(t)}\right\|_{W^{k+1,\infty}(\Gamma(t))} < C. \tag{8.1}$$

Our assumptions imply that there exists homomorphism $\phi_t^\Omega \colon L^2(\Omega_0) \to L^2(\Omega(t))$ and $\phi_t^\Gamma \colon L^2(\Gamma_0) \to L^2(\Gamma(t))$ which form compatible pairs $(L^2(\Omega(t)), \phi_t^\Omega)_{t\in[0,T]}$, $(L^2(\Gamma(t)), \phi_t^\Gamma)_{t\in[0,T]}$, $(H^1(\Omega(t)), \phi_t^\Omega)_{t\in[0,T]}$, $(H^1(\Gamma(t)), \phi_t^\Gamma)_{t\in[0,T]}$. It follows that the product spaces $\mathcal{H}(t) = L^2(\Omega(t)) \times L^2(\Gamma(t))$ and $\mathcal{V}(t) = H^1(\Omega(t)) \times H^1(\Gamma(t))$, defined for $t \in [0,T]$, form compatible pairs $(\mathcal{H}(t), \phi_t)_{t\in[0,T]}$ and $(\mathcal{V}(t), \phi_t)_{t\in[0,T]}$ for the product push-forward map $\phi_t \colon \mathcal{H}_0 \to \mathcal{H}(t)$ given by

$$\phi_t(\eta,\xi) = (\phi_t^\Omega \eta, \phi_t^\Gamma \xi) \qquad \text{for } (\eta,\xi) \in \mathcal{H}_0.$$

The product push-forward map ϕ_t defines a material derivative for pairs $(\eta,\xi) \in C^1_{\mathcal{H}}$ which can be identified as

$$\partial^\bullet(\eta,\xi) = (\partial^\bullet\eta, \partial^\bullet\xi).$$

For further information on this functional analytic setting see Alphonse et al. (2015b, Section 5.3).

In addition we will make use of the higher order spaces $\mathcal{Z}_0(t) = H^2(\Omega(t)) \times H^2(\Gamma(t))$ and $\mathcal{Z}(t) = H^{k+1}(\Omega(t)) \times H^{k+1}(\Gamma(t))$.

8.2. **Continuous problem.** We consider a weak form of (1.7).

Problem 8.1. *Given $(u_0,v_0) \in \mathcal{V}_0$, find $(u,v) \in L^2_{\mathcal{V}}$ with $\partial^\bullet(u,v) \in L^2_{\mathcal{H}}$, such that for almost every time $t \in (0,T)$ we have*

$$\begin{aligned} m(t;\partial^\bullet(u,v),\varphi) + g(t;(u,v),\varphi) + a(t;(u,v),\varphi) &= 0 \qquad \textit{for all } \varphi \in \mathcal{V}(t) \\ u(\cdot,0) = u_0, \qquad v(\cdot,0) &= v_0, \end{aligned} \tag{8.2}$$

where for $(\eta,\xi),(\chi,\rho) \in \mathcal{V}(t)$

$$m\big(t;(\eta,\xi),(\chi,\rho)\big) = \alpha \int_{\Omega(t)} \eta\chi \,\mathrm{d}x + \beta \int_{\Gamma(t)} \xi\rho \,\mathrm{d}\sigma$$

$$g\big(t;(\eta,\xi),(\chi,\rho)\big) = \alpha \int_{\Omega(t)} \eta\chi \nabla\cdot w \,\mathrm{d}x + \beta \int_{\Gamma(t)} \xi\rho \nabla_\Gamma \cdot w \,\mathrm{d}\sigma$$

$$\begin{aligned} a\big(t;(\eta,\xi),(\chi,\rho)\big) = {}& \alpha \int_{\Omega(t)} \mathscr{A}_\Omega \nabla\eta\cdot\nabla\chi + \mathscr{B}_\Omega \eta \cdot \nabla\chi + \mathscr{C}_\Omega \eta\chi \,\mathrm{d}x \\ &+ \beta \int_{\Gamma(t)} \mathscr{A}_\Gamma \nabla_\Gamma \xi \cdot \nabla\rho + \mathscr{B}_\Gamma \xi \cdot \nabla_\Gamma \rho + \mathscr{C}_\Gamma \xi\rho \,\mathrm{d}\sigma \\ &+ \int_{\Gamma(t)} (\alpha\eta - \beta\xi)(\alpha\chi - \beta\rho)\,\mathrm{d}\sigma. \end{aligned}$$

We can combine the transport formula for the surface and bulk only cases (6.2) and (7.2) for m and (6.3) and (7.3) for a to derive transport laws for these coupled bilinear forms First, for $(\eta,\xi),\varphi \in C^1_{\mathcal{H}}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} m\big(t;(\eta,\varphi),\varphi\big) = m\big(t;\partial^\bullet(\eta,\varphi),\varphi\big) + m\big(t;(\eta,\varphi),\partial^\bullet\varphi\big) + g\big(t;(\eta,\varphi),\varphi\big) \tag{8.3}$$

and for $(\eta,\xi),\varphi \in C^1_{\mathcal{V}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} a\big(t;(\eta,\varphi),\varphi\big) = a\big(t;\partial^\bullet(\eta,\varphi),\varphi\big) + a\big(t;(\eta,\varphi),\partial^\bullet\varphi\big) + b\big(t;(\eta,\varphi),\varphi\big), \tag{8.4}$$

where $b(t;\cdot,\cdot)\colon \mathcal{V}(t)\times\mathcal{V}(t) \to \mathbb{R}$ is given for $(\eta,\xi),(\chi,\rho) \in \mathcal{V}$ by

$$\begin{aligned} b\big(t;(\eta,\xi),(\chi,\rho)\big) = {}& \alpha \int_{\Omega(t)} \mathcal{B}(w,\mathscr{A}_\Omega)\nabla\eta\cdot\nabla\chi + \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}_\Omega)\eta\cdot\nabla\chi \\ &\qquad + (\partial^\bullet \mathscr{C}_\Omega + \mathscr{C}_\Omega \nabla\cdot w)\eta\chi \,\mathrm{d}x \\ &+ \beta \int_{\Gamma(t)} \mathcal{B}(w,\mathscr{A}_\Gamma)\nabla_\Gamma\xi\cdot\nabla_\Gamma\rho + \mathcal{B}_{\mathrm{adv}}(w,\mathscr{B}_\Gamma)\xi\cdot\nabla_\Gamma\rho \\ &\qquad + (\partial^\bullet \mathscr{C}_\Gamma + \mathscr{C}_\Gamma \nabla_\Gamma\cdot w)\xi\rho \,\mathrm{d}\sigma \\ &+ \int_{\Gamma(t)} (\alpha\eta - \beta\xi)(\alpha\chi - \beta\rho)\nabla_\Gamma\cdot w \,\mathrm{d}\sigma. \end{aligned}$$

We also have the estimates that there exists a constant $c > 0$ such that for all $t \in (0,T)$ we have

$$\big|g\big(t;(\eta,\chi),\varphi\big)\big| \le c\,\|(\eta,\chi)\|_{\mathcal{H}(t)}\,\|\varphi\|_{\mathcal{H}(t)} \qquad \text{for all } (\eta,\chi),\varphi \in \mathcal{H}(t) \tag{8.5}$$

$$\big|b\big(t;(\eta,\chi),\varphi\big)\big| \le c\,\|(\eta,\chi)\|_{\mathcal{V}(t)}\,\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for all } (\eta,\chi),\varphi \in \mathcal{V}(t). \tag{8.6}$$

Here we have applied the assumption (8.1) on w.

Theorem 8.2. *There exists a unique solution pair* $(u,v) \in L^2_{\mathcal{V}}$ *with* $\partial^\bullet(u,v) \in L^2_{\mathcal{H}}$*, to Problem 8.1 which satisfies the stability bound:*

$$\int_0^T \|(u,v)\|^2_{\mathcal{V}(t)} + \|\partial^\bullet(u,v)\|^2_{\mathcal{H}(t)} \,\mathrm{d}t \le c\,\|u_0\|^2_{\mathcal{V}_0}. \tag{8.7}$$

Proof. We again apply the abstract theory of Theorem 2.9 and check the assumptions.

It is clear that (M1) and (M2) hold since $m(t;\cdot,\cdot)$ is equal to the $\mathcal{H}(t)$-inner product. The assumptions (G1) and (G2) are shown in (8.3) and (8.5). We know that the map $t \mapsto a(t;\cdot,\cdot)$ is differentiable hence measurable which shows (A1). The coercivity (A2) and boundedness (A3) of a follow from standard arguments since the extra cross term is clearly positive (see also (Elliott and Ranner, 2013, Thm. 3.2)). The existence of the bilinear form b (B1) has been shown in (8.4) and the estimate (B2) is shown in (8.6). ☐

8.3. **Finite element method.** Fix $k \geq 1$. In order to define our computational method we use the construction of the isoparametric domain of order k used in Section 7.2. This defines a discrete computational domain $\{\Omega_h(t)\}_{t\in[0,T]}$ equipped with an evolving conforming subdivision $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ over which we can define an evolving (flat) surface finite element space $\{\mathcal{V}_h^\Omega(t)\}$ consisting of Lagrange finite elements of order k. We will assume that $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ is a uniformly quasi-uniform subdivision.

For $t \in [0,T]$, we write $\Gamma_h(t) = \partial\Omega_h(t)$ and $\mathscr{T}_h^\Gamma(t)$ for the boundary faces of $\mathscr{T}_h(t)$:

$$\mathscr{T}_h^\Gamma(t) := \{K(t) \cap \Gamma_h(t) : K(t) \in \mathscr{T}_h(t)\}.$$

We note that $\{\mathscr{T}_h^\Gamma(t)\}_{t\in[0,T]}$ is also an evolving conforming subdivision which we assume is uniformly quasi-uniform. In fact, this is the construction we have previously used for an evolving surface in Section 6. We define an evolving surface finite element space $\{\mathcal{V}_h^\Gamma(t)\}$ consisting of Lagrange finite elements of order k over $\{\mathscr{T}_h^\Gamma(t)\}_{t\in[0,T]}$. We define $\mathcal{V}_h(t) = \mathcal{V}_h^\Omega(t) \times \mathcal{V}_h^\Gamma(t)$.

We equip the product finite element space $\mathcal{V}_h(t)$ with the norms:

$$\|(\chi_h,\rho_h)\|_{\mathcal{V}_h(t)} := \Big(\|\chi_h\|^2_{H^1(\mathscr{T}_h(t))} + \|\rho_h\|^2_{H^1(\mathscr{T}_h^\Gamma(t))}\Big)^{1/2}$$
$$\|(\chi_h,\rho_h)\|_{\mathcal{H}_h(t)} := \Big(\|\chi_h\|^2_{L^2(\Omega_h(t))} + \|\rho_h\|^2_{L^2(\Gamma_h(t))}\Big)^{1/2}.$$

The previous constructions define a flow map $\Phi_t^h \colon \bar{\Omega}_{h,0} \to \bar{\Omega}_h(t)$ and discrete velocity W_h. Since we have assumed that $\{\mathscr{T}_h(t)\}_{t\in[0,T]}$ and $\{\mathscr{T}_h^\Gamma(t)\}_{t\in[0,T]}$ are both uniformly quasi-uniform, we can define a material derivative for functions $(\eta_h,\xi_h) \in C^1_{\mathcal{V}_h}$ which can be identified as

$$\partial_h^\bullet(\eta_h,\xi_h) = (\partial_h^\bullet\eta_h, \partial_h^\bullet\xi_h).$$

The finite element method is based on the variation form (2.6) of Problem 8.1. We introduce element-wise smooth $(n+1)\times(n+1)$-diffusion tensors $\mathscr{A}_{\Omega_h}$ and $\mathscr{A}_{\Gamma_h}$, element-wise smooth vector fields $\mathscr{B}_{\Omega_h}$ and $\mathscr{B}_{\Gamma_h}$ and an element-wise smooth scalar fields $\mathscr{C}_{\Omega_h}$ and $\mathscr{C}_{\Gamma_h}$. We assume that $\mathscr{A}_{\Omega_h}$ and $\mathscr{A}_{\Gamma_h}$ are uniformly positive definite on the element-wise tangent spaces to $\Omega_h(t)$ and $\Gamma_h(t)$, respectively, and that $\mathscr{B}_{\Gamma_h}$ is a element-wise tangential vector field on $\Gamma_h(t)$. We make the further assumption that

$$\sup_{h\in(0,h_0)}\sup_{t\in[0,T]}\Big(\big\|\mathscr{A}_{\Omega_h}\big\|_{L^\infty(\Omega_h(t))} + \big\|\mathscr{B}_{\Omega_h}\big\|_{L^\infty(\Omega_h(t))} + \big\|\mathscr{C}_{\Omega_h}\big\|_{L^\infty(\Omega_h(t))}\Big)$$
$$+\Big(\big\|\mathscr{A}_{\Gamma_h}\big\|_{L^\infty(\Gamma_h(t))} + \big\|\mathscr{B}_{\Gamma_h}\big\|_{L^\infty(\Gamma_h(t))} + \big\|\mathscr{C}_{\Gamma_h}\big\|_{L^\infty(\Gamma_h(t))}\Big) \leq C.$$

Problem 8.3. *Given $(U_{h,0},V_{h,0}) \in \mathcal{V}_{h,0}$, find $(U_h,V_h) \in C^1_{\mathcal{V}_h}$ such that*
(8.8)
$$\frac{\mathrm{d}}{\mathrm{d}t} m_h\big(t;(U_h,V_h),\phi_h\big) + a_h\big(t;(U_h,V_h),\phi_h\big) = m_h\big(t;(U_h,V_h),\partial_h^\bullet\phi_h\big) \quad \textit{for all } \phi_h \in C^1_{\mathcal{V}_h}$$
$$U_h(\cdot,0) = U_{h,0}, V_h(\cdot,0) = V_{h,0},$$

where for $(\eta_h,\xi_h),(\chi_h,\rho_h) \in \mathcal{V}_h(t)$*, we define*

$$m_h\big(t;(\eta_h,\xi_h),(\chi_h,\rho_h)\big) = \alpha \int_{\Omega_h(t)} \eta_h \chi_h \,\mathrm{d}x + \beta \int_{\Gamma_h(t)} \xi_h \rho_h \,\mathrm{d}\sigma_h$$

$$\begin{aligned} a_h\big(t;(\eta_h,\xi_h),(\chi_h,\rho_h)\big) &= \alpha \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathscr{A}_K \nabla\eta_h \cdot \nabla\chi_h + \mathscr{B}_K \eta_h \cdot \nabla\chi_h + \mathscr{C}_K \eta\chi_h \,\mathrm{d}x \\ &\quad + \beta \sum_{E(t)\in\mathscr{T}_h^\Gamma(t)} \int_{E(t)} \mathscr{A}_E \nabla_E \xi_h \cdot \nabla\rho_h + \mathscr{B}_E \xi_h \cdot \nabla_E \rho_h + \mathscr{C}_E \xi_h \rho_h \,\mathrm{d}\sigma_h \\ &\quad + \int_{\Gamma_h} (\alpha\eta_h - \beta\xi_h)(\alpha\chi_h - \beta\rho_h)\,\mathrm{d}\sigma_h. \end{aligned}$$

8.4. **Stability.** To show existence and stability for our discrete scheme we require two further lemmas.

Lemma 8.4. *The discrete velocity* W_h *of the discrete evolving domain* $\{\bar{\Omega}_h(t)\}$ *is uniformly bounded in* $W^{1,\infty}(\mathscr{T}_h(t))$ *and* $W^{1,\infty}(\mathscr{T}_h^\Gamma(t))$*. That is there exists a constant* $c > 0$ *such that for all* $h \in (0,h_0)$*,*

$$\sup_{t\in[0,T]} \Big(\|W_h\|_{W^{1,\infty}(\mathscr{T}_h(t))} + \big\|W_h|_{\Gamma_h}\big\|_{W^{1,\infty}(\mathscr{T}_h^\Gamma(t))} \Big) \le c. \tag{8.9}$$

Proof. We simply use the results (6.12) and (7.10) using the assumption (8.1). □

We also have discrete transport formula from the bulk and surface cases:

Lemma 8.5. *There exists bilinear forms* $g_h(t;\cdot,\cdot), b_h(t;\cdot,\cdot)\colon \mathcal{V}_h(t)\times\mathcal{V}_h(t)\to\mathbb{R}$ *such that for all* $(Z_h,Y_h),(\chi_h,\rho_h)\in\mathcal{C}^1_{\mathcal{V}_h}$ *we have*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} m_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big) \\ &= m_h\big(t;\partial_h^\bullet(Z_h,Y_h),(\chi_h,\rho_h)\big) + m_h\big(t;(Z_h,Y_h),\partial_h^\bullet(\chi_h,\rho_h)\big) \\ &\quad + g_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big) \end{aligned} \tag{8.10}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} a_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big) \\ &= a_h\big(t;\partial_h^\bullet(Z_h,Y_h),(\chi_h,\rho_h)\big) + a_h\big(t;(Z_h,Y_h),\partial_h^\bullet(\chi_h,\rho_h)\big) \\ &\quad + b_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big), \end{aligned} \tag{8.11}$$

where

$$g_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big) = \alpha\int_{\Omega_h(t)} Z_h\chi_h \nabla\cdot W_h \,\mathrm{d}x + \beta\int_{\Gamma_h(t)} Y_h\rho_h \nabla_{\Gamma_h}\cdot W_h \,\mathrm{d}\sigma_h,$$

and

$$\begin{aligned} &b_h\big(t;(Z_h,Y_h),(\chi_h,\rho_h)\big) \\ &= \sum_{K(t)\in\mathscr{T}_h(t)} \int_{K(t)} \mathcal{B}_h(W_h,\mathscr{A}_{\Omega_h})\nabla Z_h\cdot\nabla\chi_h + \mathcal{B}_{\mathrm{adv},h}(W_h,\mathscr{B}_{\Omega_h})Z_h\cdot\nabla\chi_h \\ &\qquad\qquad + (\partial_h^\bullet \mathscr{C}_{\Omega_h} + \mathscr{C}_{\Omega_h}\nabla\cdot W_h)Z_h\chi_h\,\mathrm{d}x \\ &+ \sum_{E(t)\in\mathscr{T}_h^\Gamma(t)} \int_{E(t)} \mathcal{B}_h(W_h,\mathscr{A}_{\Gamma_h})\nabla_E Y_h\cdot\nabla_E\rho_h + \mathcal{B}_{\mathrm{adv},h}(W_h,\mathscr{B}_{\Gamma_h})Y_h\cdot\nabla_E\rho_h \\ &\qquad\qquad + (\partial_h^\bullet \mathscr{C}_{\Gamma_h} + \mathscr{C}_{\Gamma_h}\nabla_E\cdot W_h)Y_h\rho_h\,\mathrm{d}\sigma_h. \end{aligned}$$

Theorem 8.6. *There exists a unique solution pair (U_h, V_h) of the finite element scheme (Problem 8.3) which satisfies the stability bound*

$$\sup_{t \in (0,T)} \|(U_h, V_h)\|^2_{\mathcal{H}_h(t)} + \int_0^T \|(U_h, V_h)\|^2_{\mathcal{V}_h(t)} \, \mathrm{d}t \leq c \left\|(U_{h,0}, V_{h,0})\right\|^2_{\mathcal{H}_h(t)}. \tag{8.12}$$

Proof. We apply the abstract result of Theorem 3.3 and check the assumptions.

The assumptions on m_h, (M_h1) and (M_h2), follow directly since m_h is equal to the $\mathcal{H}_h(t)$ inner-product. The estimates on a_h, (A_h2) and (A_h3) follow in the same manner as Theorem 8.2. The transport formulae and estimates for g_h and b_h, (G_h1), (G_h2) (B_h1) and (B_h2), are shown in Lemma 8.5. □

8.5. **Error analysis.** We have already constructed a bijection between the computational domain $\bar{\Omega}_h(t)$ and the continuous domain $\bar{\Omega}(t)$. In Section 7.4, for each $t \in [0,T]$, we constructed element-wise a bijection $\Lambda_h(\cdot,t) \colon \bar{\Omega}_h(t) \to \bar{\Omega}(t)$. Furthermore, we note that the restriction of the lifting operator to $\Gamma_h(t)$, $\Lambda_h(\cdot,t)|_{\Gamma_h(t)}$, is simply the normal projection operator which is the lifting operator used in Section 6.4.

For $t \in [0,T]$ and a function pair $(\eta_h, \xi_h) \in \mathcal{V}_h(t)$, we define the lift $(\eta_h, \xi_h)^\ell \colon \Omega(t) \times \Gamma(t) \to \mathbb{R}^2$ by

$$(\eta_h, \xi_h)^\ell (\Lambda_h(x,t), p(y,t)) = \big(\eta_h(x), \xi_h(y)\big) \qquad \text{for } x \in \Omega_h(t), y \in \Gamma_h(t).$$

We will often write $(\eta_h, \xi_h)^\ell = (\eta_h^\ell, \xi_h^\ell)$ to signify that the lifting process is simply a combination the previous lifts for the surface and bulk components.

We will also make use of an inverse lift for continuous functions on $\Omega(t) \times \Gamma(t)$. For $(\eta, \xi) \in C(\Omega(t)) \times C(\Gamma(t))$, we define the inverse lift of (η, ξ), denoted by $(\eta, \xi)^{-\ell}$ by

$$(\eta, \xi)^{-\ell}(x,y) = \big(\eta(\Lambda_h(x,t)), \xi_h(p(x,t))\big) \qquad \text{for } x \in \Omega_h(t), y \in \Gamma_h(t).$$

Lemma 8.7. *Let $(\eta_h, \xi_h) \in \mathcal{V}_h(t)$ and denote their lift by $(\eta_h, \xi_h)^\ell$. Then there exists constants $c_1, c_2 > 0$ such that*

$$c_1 \left\|(\eta_h, \xi_h)^\ell\right\|_{\mathcal{H}(t)} \leq \|(\eta_h, \xi_h)\|_{\mathcal{H}_h(t)} \leq c_2 \left\|(\eta_h, \xi_h)^\ell\right\|_{\mathcal{H}(t)} \tag{8.13}$$

$$c_1 \left\|(\eta_h, \xi_h)^\ell\right\|_{\mathcal{V}(t)} \leq \|(\eta_h, \xi_h)\|_{\mathcal{V}_h(t)} \leq c_2 \left\|(\eta_h, \xi_h)^\ell\right\|_{\mathcal{V}(t)}. \tag{8.14}$$

Proof. We simply combine the results of Lemma 7.10 and Lemma 6.10. □

For $t \in [0,T]$, we define $\mathcal{V}_h^\ell(t)$ to be a space of lifted finite element functions given by

$$\mathcal{V}_h^\ell(t) := \{(\eta_h, \xi_h)^\ell : (\eta_h, \xi_h) \in \mathcal{V}_h(t)\}.$$

This spaces is equipped with the follow approximation property:

Lemma 8.8 (Approximation property)**.** *For $(\eta, \xi) \in C(\Omega(t)) \times C(\Gamma(t))$ the Lagrangian interpolation operator $I_h(\eta, \xi)$ is well defined. Furthermore, the following bounds hold for a constant $c > 0$ for all $h \in (0, h_0)$ and $t \in [0,T]$:*

$$\|(\eta, \xi) - I_h(\eta, \xi)\|_{\mathcal{H}(t)} + h \|(\eta, \xi) - I_h(\eta, \xi)\|_{\mathcal{V}(t)} \leq c h^{k+1} \quad \text{for } (\eta, \xi) \in \mathcal{Z}(t) \tag{8.15}$$

$$\|(\eta, \xi) - I_h(\eta, \xi)\|_{\mathcal{H}(t)} + h \|(\eta, \xi) - I_h(\eta, \xi)\|_{\mathcal{V}(t)} \leq c h^2 \quad \text{for } (\eta, \xi) \in \mathcal{Z}_0(t). \tag{8.16}$$

Proof. The proof follows by combining the result of Lemma 6.11 and Lemma 7.11. □

We again use the lift to define the an evolving lifted triangulation. For each $t \in [0,T]$ and $h \in (0,h_0)$, we define

$$\mathscr{T}_h^\ell(t) := \{K^\ell(t) : K(t) \in \mathscr{T}_h(t)\}$$
$$(\mathscr{T}_h^\Gamma)^\ell(t) := \{E^\ell(t) : E(t) \in \mathscr{T}_h^\Gamma(t)\}.$$

The edges of these curvilinear-simplicies evolving with a velocity w_h which can be characterised in the same way as the bulk (7.22) and surface (6.20) cases (when the appropriate restrictions are made). Equivalently, this defines a flow $\Phi_t^\ell \colon \bar{\Omega}_0 \to \bar{\Omega}(t)$ which is the map given by

$$\Phi_t^\ell(y_0) = Y(t) \qquad \text{where } Y(t) \text{ satisfies } \frac{\mathrm{d}}{\mathrm{d}t} Y(t) = w_h(Y(t),t), Y(0) = y_0.$$

In turn, this flow defines a push-forward map ϕ_t^ℓ on $\{\mathcal{H}(t)\}_{t\in[0,T]}$ given by

$$\phi_t^\ell(\eta,\xi)(x,y) = \big(\eta(\Phi_{-t}^\ell(x)), \xi(\Phi_{-t}^\ell(y))\big) \qquad x \in \Omega(t), y \in \Gamma(t), (\eta,\xi) \in \mathcal{H}_0.$$

Lemma 8.9. *The pairs $\{\mathcal{H}(t), \phi_t^\ell\}_{t\in[0,T]}$ and $\{\mathcal{V}(t), \phi_t^\ell\}_{t\in[0,T]}$ are compatible hence we may define a material derivative $\partial_h^\bullet(\eta,\xi)$ for $(\eta,\xi) \in C^1_{\mathcal{H}}$ and the following transport formulae hold. There exists a bilinear forms $\widetilde{g}_h(t;\cdot,\cdot) \colon \mathcal{H}(t)\times\mathcal{H}(t) \to \mathbb{R}$ and $\widetilde{b}_h(t;\cdot,\cdot) \to \mathbb{R}$ such that*

(8.17)
$$\frac{\mathrm{d}}{\mathrm{d}t} m(t;(\eta,\xi),\varphi) = m(t;\partial_h^\bullet(\eta,\xi),\varphi) + m(t;(\eta,\xi),\partial_h^\bullet\varphi) + \widetilde{g}_h(t;\eta,\varphi) \quad \text{for } (\eta,\xi),\varphi \in C^1_{\mathcal{H}}$$

(8.18)
$$\frac{\mathrm{d}}{\mathrm{d}t} a(t;(\eta,\xi),\varphi) = a(t;\partial_h^\bullet(\eta,\xi),\varphi) + a(t;(\eta,\xi),\partial_h^\bullet\varphi) + \widetilde{b}_h(t;\eta,\varphi) \quad \text{for } (\eta,\xi),\varphi \in C^1_{\mathcal{V}}.$$

Furthermore, there exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $h \in (0,h_0)$ we have

$$|\widetilde{g}_h(t;(\eta,\xi),\varphi)| \le c\|(\eta,\xi)\|_{\mathcal{H}(t)}\|\varphi\|_{\mathcal{H}(t)} \qquad \text{for all } (\eta,\xi),\varphi \in \mathcal{H}(t) \tag{8.19}$$

$$\left|\widetilde{b}_h(t;(\eta,\xi),\varphi)\right| \le c\|(\eta,\xi)\|_{\mathcal{V}(t)}\|\varphi\|_{\mathcal{V}(t)} \qquad \text{for all } (\eta,\xi),\varphi \in \mathcal{V}(t). \tag{8.20}$$

Proof. We combine the results of Lemma 6.13 and Lemma 7.12. □

The geometric perturbation results now follow directly by combining the appropriate results from Sections 6.4 and 7.4.

Lemma 8.10. *We have the estimates*

$$|w - w_h|_{L^\infty(\Omega(t))} + h|\nabla(w - w_h)|_{L^\infty(\Omega(t))} \tag{8.21}$$

$$|w - w_h|_{L^\infty(\Gamma(t))} + h|\nabla_\Gamma(w - w_h)|_{L^\infty(\Gamma(t))} \le ch^{k+1}. \tag{8.22}$$

In particular, this implies

$$\|\partial^\bullet(\eta,\xi) - \partial_h^\bullet(\eta,\xi)\|_{\mathcal{H}(t)} \le ch^{k+1} \qquad \text{for } (\eta,\xi) \in C^1_{\mathcal{V}} \tag{8.23a}$$

$$\|\partial^\bullet(\eta,\xi) - \partial_h^\bullet(\eta,\xi)\|_{\mathcal{V}(t)} \le ch^{k} \qquad \text{for } (\eta,\xi) \in C^1_{\mathcal{Z}_0}. \tag{8.23b}$$

Proof. We combine the results of Lemmas 6.12, 6.18, and 7.14. □

Lemma 8.11. *There exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $h \in (0,h_0)$ the following holds for all $\mathcal{V}_h, \phi_h \in \mathcal{V}_h(t) + \mathcal{Z}_0^{-\ell}(t)$ with lifts $v_h = V_h^\ell, \varphi_h = \phi_h^\ell \in \mathcal{V}_h^\ell(t) + \mathcal{Z}_0(t)$:*

$$|m(t;v_h,\varphi_h) - m_h(t;V_h,\phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{8.24}$$

$$|\widetilde{g}_h(t;v_h,\varphi_h) - g_h(t;V_h,\phi_h)| \le ch^{k+1} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{8.25}$$

$$|a(t;v_h,\varphi_h) - a_h(t;V_h,\phi_h)| \le ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{8.26}$$

$$\left|\widetilde{b}_h(t;v_h,\varphi_h) - b_h(t;V_h,\phi_h)\right| \le ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)} \tag{8.27}$$

$$\left|\widetilde{b}_h(t;v_h,\varphi_h) - b(t;v_h,\varphi_h)\right| \le ch^{k} \|v_h\|_{\mathcal{V}(t)} \|\varphi_h\|_{\mathcal{V}(t)}. \tag{8.28}$$

For $\eta, \varphi \in \mathcal{Z}_0(t)$ with inverse lifts $\eta^{-\ell}, \varphi^{-\ell}$, we have

$$\left|a(t;\eta,\varphi) - a_h(t;\eta^{-\ell},\varphi^{-\ell})\right| \le ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \tag{8.29}$$

$$\left|\widetilde{b}_h(t;\eta,\varphi) - b_h(t;\eta^{-\ell},\varphi^{-\ell})\right| \le ch^{k+1} \|\eta\|_{\mathcal{Z}_0(t)} \|\varphi\|_{\mathcal{Z}_0(t)} \tag{8.30}$$

$$\left|a(t;\partial_h^\bullet\eta,\varphi) - a_h(t;\partial_h^\bullet\eta^{-\ell},\varphi^{-\ell})\right| \le ch^{k+1} (\|\eta\|_{\mathcal{Z}_0(t)} + \|\partial^\bullet\eta\|_{\mathcal{Z}_0(t)}) \|\varphi\|_{\mathcal{Z}_0(t)} \tag{8.31}$$

Proof. We combine the results of Lemma 6.17 and 7.15. □

Theorem 8.12. *Let $\mathscr{A}_{\Omega_h} = \mathscr{A}_\Omega^{-\ell}$, $\mathcal{B}_{\Omega_h} = \mathcal{B}_\Omega^{-\ell}$, $\mathcal{C}_{\Omega_h} = \mathcal{C}_\Omega^{-\ell}$, $\mathscr{A}_{\Gamma_h} = \mathscr{A}_\Gamma^{-\ell}$, $\mathcal{B}_{\Gamma_h} = \mathcal{B}_\Gamma^{-\ell}$, $\mathcal{C}_{\Gamma_h} = \mathcal{C}_\Gamma^{-\ell}$. Let $(u,v) \in L^2_{\mathcal{V}}$ be the solution of (8.2) which we assume satisfies the regularity bound*

$$\sup_{t\in(0,T)} \|(u,v)\|^2_{\mathcal{Z}(t)} + \int_0^T \|\partial^\bullet(u,v)\|^2_{\mathcal{Z}(t)} \,\mathrm{d}t \le C_{u,v}. \tag{8.32}$$

Let $(U_h,V_h) \in C^1_{\mathcal{V}_h}$, be the solution of the finite element scheme (8.8) and write $(u_h,v_h) = (U_h,V_h)^\ell$. Then we have the following error estimate

$$\sup_{t\in(0,T)} \|(u,v)-(u_h,v_h)\|^2_{\mathcal{H}(t)} + h^2 \int_0^T \|(u,v)-(u_h,v_h)\|^2_{\mathcal{V}(t)} \,\mathrm{d}t \le c\left\|(u_0,v_0)-(u_{h,0},v_{h,0})\right\|^2_{\mathcal{H}(t)} + ch^{2k+2}C_{u,v}. \tag{8.33}$$

Proof. We apply abstract Theorem 3.8 and check the assumptions.

We know the lift is stable from Lemma (8.7). The existence and boundedness of $\widetilde{g}_h$ and $\widetilde{b}_h^\kappa$ are dealt with in Lemma 8.9. The interpolation properties (I1) and (I2) are shown in Lemma 8.8. The geometric perturbation estimates (P1)–(P8) are shown in Lemmas 8.11 and 8.10. □

9. Numerical results

The above finite element methods were implemented using DUNE. We discretize in time using an implicit Euler time stepping scheme. The time step τ is scaled so that the optimal error scales are recovered. At each time step we solve the full system using the generalized minimal residual method,

The code produced to run these computations is available at

https://github.com/tranner/dune-evolving-domains

9.1. **Test geometry.** Let $T = 1$, and $t \in [0,T]$. For $t \geq 0$, we define $\Omega(t)$ via a parametrisation $G\colon \Omega_0 \times \mathbb{R}_+ \to \mathbb{R}$, for $\Omega_0 = B(0,1) \subset \mathbb{R}^3$ the unit ball in three-dimensions. The parametrisation $G\colon \bar{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^3$ is given by

$$G(x,t) = \left(a(t)^{1/2}x_1, x_2, x_3\right), \qquad a(t) = 1 + \frac{1}{4}\sin(t),$$

with velocity field w given by

$$w(x,t) = \left(\frac{\cos(t)x_1}{8(1+1/4\sin(t))}, 0, 0\right) \qquad \text{for } x \in \bar{\Omega}(t).$$

The geometry is the same for each problem, which corresponds to an ellipsoidal domain growing along a single axis, but we solve in and on different parts of the domain.

For each test problem, for each iteration we complete an appropriate number of bisectional refinements in order to approximately half the mesh size h and scale the time step τ to recover the optimal order of convergence – i.e. $\tau_j = \tau_0 2^{-(k+1)j}$. We show the error in an L^2-norm at the final time. The experimental order of convergence (eoc) at level j is computed by

$$(\text{eoc})_j = \log(E_j/E_{j-1})/\log(h_j/h_{j-1}).$$

Errors in an H^1-norm demonstrate an order of convergence less and are not listed here.

9.2. **Problem on a closed surface (1.3).** We set the parameters in the equation as $\mathscr{A} = (1+x_1^2)\mathrm{Id}$, $\mathscr{B} = (1,2,0) - (1,2,0)\cdot \nu\nu$, $\mathscr{C} = \sin(x_1 x_2)$ and compute an additional a right hand side in (1.3a) and take appropriate initial data so that the solution is given by

$$u(x,t) = \sin(t)x_2x_3 \qquad\qquad \text{for } x \in \Gamma(t).$$

We compute with $k = 2,3$. The results are shown in Tables 1 and 2.

h	τ	$L^2(\Gamma(T))$ error	(eoc)
$8.31246 \cdot 10^{-1}$	1.00000	$9.83996 \cdot 10^{-2}$	—
$4.40053 \cdot 10^{-1}$	$1.25000 \cdot 10^{-1}$	$1.47435 \cdot 10^{-2}$	2.98450
$2.22895 \cdot 10^{-1}$	$1.56250 \cdot 10^{-2}$	$1.99237 \cdot 10^{-3}$	2.94251
$1.11969 \cdot 10^{-1}$	$1.95312 \cdot 10^{-3}$	$2.50039 \cdot 10^{-4}$	3.01456
$5.60891 \cdot 10^{-2}$	$2.44141 \cdot 10^{-4}$	$3.12365 \cdot 10^{-5}$	3.00895

TABLE 1. $k = 2$

h	τ	$L^2(\Gamma(T))$ error	(eoc)
$8.31246 \cdot 10^{-1}$	1.00000	$9.88086 \cdot 10^{-2}$	—
$4.40053 \cdot 10^{-1}$	$6.25000 \cdot 10^{-2}$	$7.60635 \cdot 10^{-3}$	4.03157
$2.22895 \cdot 10^{-1}$	$3.90625 \cdot 10^{-3}$	$4.92316 \cdot 10^{-4}$	4.02476
$1.11969 \cdot 10^{-1}$	$2.44141 \cdot 10^{-4}$	$3.08257 \cdot 10^{-5}$	4.02448
$5.60891 \cdot 10^{-2}$	$1.52588 \cdot 10^{-5}$	$1.89574 \cdot 10^{-6}$	4.03416

TABLE 2. $k = 3$

9.3. **Problem on in a bulk domain (1.5).** We set the parameters in the equation as $\mathscr{A} = (1+x_1^2)\mathrm{Id}$, $\mathscr{B} = (1,2,0)$, $\mathscr{C} = \cos(x_1x_2)$ and compute additional a right hand sides in (1.5a) and (1.5b) and take appropriate initial data so that the solution is given by

$$u(x,t) = \sin(t)\cos(\pi x_1)\cos(\pi x_2) \qquad \text{for } x \in \Omega(t).$$

We compute with $k = 1,2$. The results are shown in Tables 3 and 4.

h	τ	$L^2(\Omega(T))$ error	(eoc)
1.10017	1.00000	$7.54412 \cdot 10^{-2}$	—
$8.82662 \cdot 10^{-1}$	$2.50000 \cdot 10^{-1}$	$1.72380 \cdot 10^{-1}$	−3.75139
$5.23405 \cdot 10^{-1}$	$6.25000 \cdot 10^{-2}$	$1.07326 \cdot 10^{-1}$	0.90670
$2.79882 \cdot 10^{-1}$	$1.56250 \cdot 10^{-2}$	$3.17823 \cdot 10^{-2}$	1.94407
$1.44128 \cdot 10^{-1}$	$3.90625 \cdot 10^{-3}$	$8.34529 \cdot 10^{-3}$	2.01489

TABLE 3. $k = 1$

h	τ	$L^2(\Omega(T))$ error	(eoc)
1.10017	1.00000	$4.46630 \cdot 10^{-2}$	—
$8.82662 \cdot 10^{-1}$	$1.25000 \cdot 10^{-1}$	$4.44526 \cdot 10^{-2}$	0.02144
$5.23405 \cdot 10^{-1}$	$1.56250 \cdot 10^{-2}$	$7.59648 \cdot 10^{-3}$	3.38076
$2.79882 \cdot 10^{-1}$	$1.95312 \cdot 10^{-3}$	$1.05698 \cdot 10^{-3}$	3.15065
$1.44128 \cdot 10^{-1}$	$2.44141 \cdot 10^{-4}$	$1.38589 \cdot 10^{-4}$	3.06126

TABLE 4. $k = 2$

9.4. **Problem on a coupled bulk-surface domain (1.7).** We set the parameters in the equation as $\mathscr{A}_X = \mathrm{Id}$, $\mathscr{B}_X = (0,0,0)$, $\mathscr{C}_X = 0$, for $X = \Omega$ and Γ, and $\alpha = \beta = 1$, and compute additional a right hand sides in (1.7a), (1.7b) and (1.7c) and take appropriate initial data so that the solution is given by

$$u(x,t) = \sin(t)x_1x_2 \qquad \text{for } x \in \Omega(t)$$
$$v(x,t) = \sin(t)x_2x_3 \qquad \text{for } x \in \Gamma(t).$$

We compute with $k = 1,2$. The results are shown in Tables 5 and 6.

h	τ	$L^2(\Omega(T))$ error	(eoc)	$L^2(\Gamma(T))$ error	(eoc)
1.10017	1.00000	$1.40014 \cdot 10^{-2}$	—	$7.41054 \cdot 10^{-2}$	—
$8.82662 \cdot 10^{-1}$	$2.50000 \cdot 10^{-1}$	$2.61297 \cdot 10^{-2}$	−2.83240	$4.53161 \cdot 10^{-2}$	2.23275
$5.23405 \cdot 10^{-1}$	$6.25000 \cdot 10^{-2}$	$9.52446 \cdot 10^{-3}$	1.93118	$1.58725 \cdot 10^{-2}$	2.00746
$2.79882 \cdot 10^{-1}$	$1.56250 \cdot 10^{-2}$	$2.61552 \cdot 10^{-3}$	2.06458	$4.25452 \cdot 10^{-3}$	2.10325
$1.44128 \cdot 10^{-1}$	$3.90625 \cdot 10^{-3}$	$6.72781 \cdot 10^{-4}$	2.04591	$1.08139 \cdot 10^{-3}$	2.06389

TABLE 5. $k = 1$

h	τ	$L^2(\Omega(T))$ error	(eoc)	$L^2(\Gamma(T))$ error	(eoc)
1.10017	1.00000	$2.44058 \cdot 10^{-2}$	—	$1.22069 \cdot 10^{-1}$	—
$8.82662 \cdot 10^{-1}$	$1.25000 \cdot 10^{-1}$	$2.92797 \cdot 10^{-3}$	9.62654	$9.65135 \cdot 10^{-3}$	11.51950
$5.23405 \cdot 10^{-1}$	$1.56250 \cdot 10^{-2}$	$4.02385 \cdot 10^{-4}$	3.79775	$1.47977 \cdot 10^{-3}$	3.58832
$2.79882 \cdot 10^{-1}$	$1.95312 \cdot 10^{-3}$	$5.08882 \cdot 10^{-5}$	3.30323	$1.87863 \cdot 10^{-4}$	3.29708
$1.44128 \cdot 10^{-1}$	$2.44141 \cdot 10^{-4}$	$6.36219 \cdot 10^{-6}$	3.13299	$2.34864 \cdot 10^{-5}$	3.13304

TABLE 6. $k = 2$

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