

EVOLVING FINITE ELEMENTS FOR ADVECTION DIFFUSION WITH AN EVOLVING INTERFACE

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ABSTRACT. The aim of this paper is to develop a numerical scheme to approximate evolving interface problems for parabolic equations based on the abstract evolving finite element framework proposed in [22]. An appropriate weak formulation of the problem is derived for the use of evolving finite elements designed to accommodate for a moving interface. Optimal order error bounds are proved for arbitrary order evolving isoparametric finite elements. The paper concludes with numerical results for a model problem verifying orders of convergence.

1. INTRODUCTION

The model studied in the paper is the following, let Ω be a stationary domain with a moving interface $\Gamma(t)$ that encloses a subdomain $\Omega_1(t)$ and let $\Omega_2(t) = \Omega \setminus \overline{\Omega_1(t)}$. We denote by ν_Γ the outward pointing normal to $\Omega_1(t)$. Let, for $i = 1$ and 2 , \mathcal{A}_i be a diffusion tensor field, \mathcal{B}_i be a vector field and \mathcal{C}_i be a scalar field, each continuous on $\Omega_i(t)$. Let f_1, f_2 and g be time-dependent functions on $\Omega_1(t)$, $\Omega_2(t)$ and $\Gamma(t)$ respectively. More precise definitions of the problem data are given in [Thm. 2.16](#). We are interested in well posedness and a suitable finite element scheme for the solutions of the following problem: Find scalar fields u_1 on the subdomain $\Omega_1(t)$ and u_2 on the subdomain $\Omega_2(t)$, which satisfy:

$$\partial_t u_i - \nabla \cdot (\mathcal{A}_i(t; x) \nabla u_i) + \mathcal{B}_i(t; x) \cdot \nabla u_i + \mathcal{C}_i(t; x) u_i = f_i(t; x) \quad \text{in } \Omega_i(t), \quad (1.1a)$$

$$u_2 = 0 \quad \text{on } \partial\Omega, \quad (1.1b)$$

$$u_1 - u_2 = 0 \quad \text{on } \Gamma(t), \quad (1.1c)$$

$$\mathcal{A}_1(t; x) \frac{\partial}{\partial \nu_\Gamma} u_1 \Big|_{\Gamma(t)} - \mathcal{A}_2(t; x) \frac{\partial}{\partial \nu_\Gamma} u_2 \Big|_{\Gamma(t)} = g(t; x) \quad \text{on } \Gamma(t), \quad (1.1d)$$

$$u_i(0) = u_i^0 \quad \text{on } \Omega_i(0). \quad (1.1e)$$

Such equations can arise as subproblems when modelling the transport and diffusion of the concentration of a dissolved chemical species in evolving spatial domains. In particular, we mention applications in fluid dynamics [1, 11, 49], materials science [10, 28] and cell biology [29, 47, 50].

There are two main difficulties concerning this problem. The first of which is the evolution of the subdomains and the second is the presence of a discontinuous jump across the interface. One common approach to moving domains is the *ALE* (Arbitrary Eulerian Lagrangian) method, see [32, 38, 48]. This involves having a parametrisation of the evolving region. The flow associated with this parametrisation could be physical or could be made to fit a specific purpose such as in [18] where the flow is chosen using knowledge of the surface velocity to construct a harmonic extension. Another common method is to use a discontinuous or immersed Galerkin method [2, 40, 51]. In this paper we propose and analyse an ALE approach using evolving finite elements on an evolving fitted mesh allowing the use of isoparametric elements that accurately approximate the boundary and result in higher order error estimates. The underlying parametrisation is assumed given.

The key contributions of this work are:

- We provide a functional analytic setting to show well posedness of the continuous problem, (1.1).
- We provide an ALE approach based on evolving isoparametric finite element spaces attached to evolving sub-domains. The evolving mesh is based on moving the Lagrange nodes with a given known smooth velocity. Achieving a higher order method requires a good initial mesh.
- We provide a robust error bound which demonstrates the error in an L^2 norm is bounded, up to a constant, by h^{k+1} , where h represents the mesh size and k is the degree of polynomials used both for the discretisation of the domain and the solution. This is the same order error as if we interpolated a known smooth solution.
- Numerical results and the simulation code are provided both to demonstrate the results and to allow others to use the implementation.

The assumption is made that we are given a global, smooth velocity field \mathbf{w} . Furthermore, the velocity field is such that moving the nodes of the mesh with the \mathbf{w} preserves the regularity of the mesh over time. The velocity may be derived from physical considerations or otherwise an arbitrary velocity constructed in order to define a well behaved numerical scheme. We do not address how to achieve such a velocity in this work. There are methods in the literature to prevent mesh deformation, which involve re-parametrising the flow responsible for the movement of the interface into a more suitable flow, see, for example, [9, 19, 20].

1.1. Outline. Sec. 2 gives a well posedness analysis of the continuous equations along with the necessary functional analysis setting. The finite element construction is in Sec. 3 and the finite element scheme is in Sec. 4. An optimal order error bound is shown in Sec. 5 under smoothness assumptions on the domain and its evolution and the solution. Sec. 6 includes a time discretisation of the finite element scheme along with numerical experiences demonstrating the error bounds are tight. The Appendix includes further details of the proof the well posedness of the continuous scheme.

2. EVOLVING SPACE FORMULATION AND WELL POSEDNESS

In this paper, c will be used as a generic constant that depends on no quantity of particular importance. We use $(\cdot, \cdot)_H$ for an inner product on a Hilbert space H and $\langle \cdot, \cdot \rangle_X$ as the dual pairing between a Banach space X and its topological dual X' .

2.1. Evolving Hilbert Spaces. We set up the necessary tools from the theory of evolving Sobolev spaces which were introduced and developed in [3–5]. We will only concern ourselves with the Hilbert case. A more general theory is developed in [3] concerning general Banach spaces. Let $I = [0, T]$ be a closed time interval and let $\{X(t)\}_{t \in I}$ be a family of Hilbert spaces equipped with norm $\|\cdot\|_{X(t)}$. Assume that there exists a linear map $\phi_t : X(0) \rightarrow X(t)$ satisfying the following properties:

- B1 The map ϕ_t is invertible for all $t \in I$ with inverse denoted by ϕ_{-t} and ϕ_0 being the identity.
- B2 There exists a constant C independent of time such that $\|\phi_t \eta\|_{X(t)} \leq C \|\eta\|_{X(0)}$, $\|\phi_{-t} \tilde{\eta}\|_{X(0)} \leq C \|\tilde{\eta}\|_{X(t)}$, for all $\eta \in X(0)$ and $\tilde{\eta} \in X(t)$, for all $t \in I$.
- B3 The map $t \mapsto \|\phi_t \eta\|_{X(t)}$ is measurable for all $\eta \in X(0)$.

Here and elsewhere we use the notation $\phi_t \eta$ to denote the map ϕ_t applied to η . If such a map ϕ_t exists then we call it the *flow map* and the pair $(X(t), \phi_t)_{t \in I}$ a *compatible pair*. Given a compatible pair, define the *Hilbert moving spaces* as:

$$L_X^2 := \left\{ \eta : I \rightarrow \bigcup_{t \in I} X(t) \times \{t\}, \quad t \mapsto (\hat{\eta}(t), t) \mid \phi_{-t} \hat{\eta}(t) \in L^2(I; X(0)) \right\}, \quad (2.1)$$

and the uniformly bounded equivalent:

$$L_X^\infty := \left\{ \eta : I \rightarrow \bigcup_{t \in I} X(t) \times \{t\}, \quad t \mapsto (\hat{\eta}(t), t) \mid \phi_{-t} \hat{\eta}(t) \in L^\infty(I; X(0)) \right\}. \quad (2.2)$$

We identify $\eta(t) = (\hat{\eta}(t), t)$ with $\hat{\eta}(t)$. The spaces L_X^p are equipped with the norm:

$$\|\eta\|_{L_X^p} := \begin{cases} \left(\int_0^T \|\eta(t)\|_{X(t)}^2 dt \right)^{\frac{1}{2}} & \text{for } p = 2, \\ \text{ess sup}_{t \in [0, T]} \|\eta(t)\|_{X(t)} & \text{for } p = \infty. \end{cases}$$

L_X^2 is indeed a Hilbert space, see [3, Thm. 3.4]. The analogues of the spaces of continuous functions and of compactly supported smooth functions are defined as:

$$C_X^k := \left\{ \eta : I \rightarrow \bigcup_{t \in I} X(t) \times \{t\}, \quad t \mapsto (\hat{\eta}(t), t) \mid \phi_{-t} \hat{\eta}(t) \in C^k(I; X(0)) \right\},$$

$$\mathcal{D}_X := \left\{ \eta : I \rightarrow \bigcup_{t \in I} X(t) \times \{t\}, \quad t \mapsto (\hat{\eta}(t), t) \mid \phi_{-t} \hat{\eta}(t) \in \mathcal{D}(I; X(0)) \right\}.$$

Remark 2.1. (1) The use of a Cartesian product \times inside the union in (2.1) rather than just taking the union of $\{X(t)\}_{t \in I}$ by itself is in order to guarantee a disjoint union which is crucial to identify the function point-wise.

(2) Note that the spaces L_X^p do not depend on the choice of the map ϕ_t .

The *strong material* derivative in the evolving Hilbert space setting is defined as follows:

$$\partial_t^\bullet \eta := \phi_t \partial_t (\phi_{-t} \eta), \quad \eta \in C_X^1.$$

Lemma 2.2 ([3], Thm. 2.4). *Given a compatible pair $(X(t), \phi_t)_{t \in I}$, the maps $\phi_t : L^2(I; X(0)) \rightarrow L_X^2$ and $\phi_{-t} : L_X^2 \rightarrow L^2(I; X(0))$ define continuous linear isomorphism to their respective spaces.*

Now assume $\{X(t)\}_{t \in I}$, $\{Y(t)\}_{t \in I}$ and $\{X^*(t)\}_{t \in I}$ are families of Hilbert spaces, with $X^*(t)$ the dual of $X(t)$ for all $t \in I$ (crucially, $X(t)$ and $X^*(t)$ are not identified). Assume further that for all $t \in I$, $X(t) \subset Y(t) \cong Y^*(t) \subset X^*(t)$ constitutes a Hilbert triple (in the sense that $X(t)$ is densely and continuously embedded into $Y(t)$ and $Y(t)$ is identified with its dual via Riesz representation). It is also assumed that there exists a map $\phi_t : Y(0) \rightarrow Y(t)$ with $\phi_t|_{X(0)} : X(0) \rightarrow X(t)$ with adjoint flow $\phi_{-t}^* : X^*(0) \rightarrow X^*(t)$,

$$\langle \phi_{-t}^* f, v \rangle_{X(t)} := \langle f, \phi_{-t} v \rangle_{X(0)}, \quad f \in X^*(0), v \in X(t),$$

such that $(X(t), \phi_t|_{X(0)})_{t \in I}$, $(Y(t), \phi_t)_{t \in I}$ and $(X^*(t), \phi_{-t}^*)_{t \in I}$ all define compatible pairs and therefore we can define the spaces L_X^2 , L_Y^2 , $L_{X^*}^2$ with their respective flows. In this case, just as for Bochner spaces, we have that $(L_X^2)^*$ is isometrically isomorphic to $L_{X^*}^2$ [3, Thm. 3.7]. Moreover, the Hilbert triple structure is preserved: $L_X^2 \subset L_Y^2 \subset L_{X^*}^2$. Note that L_Y^2 remains a Hilbert space with a natural inner product structure, see [3, Rem. 3.9]. In order to generalise the concept of a “weak time derivative” to the evolving space, we first assume the following:

D1 The map $t \mapsto \langle \phi_t w_0, \phi_t v_0 \rangle_{X(t)} = (\phi_t w_0, \phi_t v_0)_{Y(t)}$ is continuously differentiable for fixed $w_0, v_0 \in X_0$.

D2 For all $t \in I$, the map:

$$[w_0, v_0] \mapsto \frac{d}{dt} (\phi_t w_0, \phi_t v_0)_{Y(t)},$$

for $[w_0, v_0] \in X(0) \times X(0)$ is continuous.

D3 There exists a constant C_λ independent of time such that, for almost all $t \in I$ and $w_0, v_0 \in X(0)$, we have:

$$\left| \frac{d}{dt}(\phi_t w_0, \phi_t v_0)_{Y(t)} \right| \leq C_\lambda \|w_0\|_{Y(0)} \|v_0\|_{Y(0)}.$$

Definition 2.3. Let Ass. D1 to D3 hold and label:

$$\lambda(t; w, v) := \left[\frac{d}{dt}(\phi_t w_0, \phi_t v_0)_{Y(t)} \right]_{(w_0, v_0) = (\phi_{-t} w, \phi_{-t} v)}, \quad w, v \in X(t).$$

Then $\lambda(t; \cdot, \cdot) : Y(t) \times Y(t) \rightarrow \mathbb{R}$ is a continuous, symmetric, bounded and bilinear for almost all $t \in I$. We say $w \in L_X^1$ has a weak material derivative if there exists $v \in L_{X^*}^1$ such that:

$$\int_0^T (w(t), \partial_t^\bullet \eta)_{Y(t)} dt = \int_0^T \langle v(t), \eta \rangle_{X(t)} + \lambda(t; w, \eta) dt,$$

for all $\eta \in \mathcal{D}_X$. We label the weak material derivative $v = \partial_t^\bullet w$.

This definition follows all properties we expect from a weak derivative, such as being equivalent to the strong material derivative if the function is regular enough.

This allows us to define the equivalent of the Bochner solution space.

Definition 2.4. We define $W(X, Y) := \{v \in L_X^2, \partial_t^\bullet v \in L_Y^2\}$ with the norm:

$$\|v\|_{W(X, Y)}^2 := \|v\|_{L_X^2}^2 + \|\partial_t^\bullet v\|_{L_Y^2}^2$$

and the solution space $W(X, X^*) := \{v \in L_X^2, \partial_t^\bullet v \in L_{X^*}^2\}$ with the norm:

$$\|v\|_W^2 := \|v\|_{L_X^2}^2 + \|\partial_t^\bullet v\|_{L_{X^*}^2}^2.$$

Definition 2.5. The space $W(X, Y)$ is said to satisfy moving space equivalence if:

$$v \in W(X, Y) \iff \phi_{-t} v \in \mathcal{W}^{2,2}(X(0), Y(0)),$$

where $\mathcal{W}^{2,2}(X(0), Y(0)) = \{v_0 \in L^2(I; X(0)), \partial_t v \in L^2(I; Y(0))\}$.

Theorem 2.6 (The Transport Theorem). Assume $v, w \in W(X, X^*)$ and the moving space equivalence is satisfied, then, the map $t \mapsto (v, w)_{Y(t)}$ is uniformly continuous and for almost all $t \in I$, and the following holds:

$$\frac{d}{dt}(v, w)_{Y(t)} = \langle \partial_t^\bullet v, w \rangle_{X(t)} + \langle \partial_t^\bullet w, v \rangle_{X(t)} + \lambda(t; v, w).$$

Moreover, $C_Y^0 \hookrightarrow W(X, X^*)$.

See [3, Sec. 4.5] for proofs.

Lemma 2.7 (Characterisation of Material Derivative). Let the moving space equivalence hold for $W(X, X)$, then for $v \in W(X, X)$, we have $v \in C_X^0$ and there exists a function $v_0 \in \mathcal{W}^{2,2}(X(0), X(0))$ such that $v = \phi_t v_0$. Moreover, C_X^1 is dense in $W(X, X)$.

This follows from [3, Lem. 3.20]. Importantly, this implies $\partial_t^\bullet v = 0$ if and only if $v = \phi_t v_0$ for some $v_0 \in X(0)$.

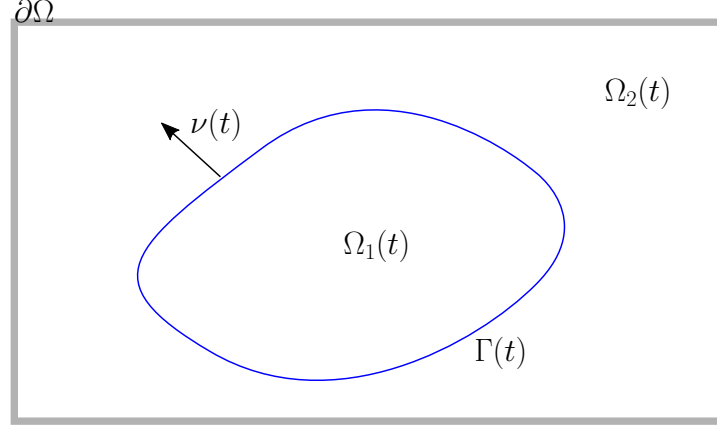


FIGURE 2.1. An example configuration of the domain.

2.2. Setting up the Domain. Let Ω be a stationary domain in \mathbb{R}^d , $d = 2, 3$, with piecewise linear boundary and let $\{\Gamma(t), t \in I\}$ be a family of closed compact connected C^{2+k} ($k \geq 0$) hypersurfaces with $\Gamma(t) \subset \Omega$. Let $\Omega_1(t)$ be a domain in Ω without boundary $\partial\Omega_1(t) = \Gamma(t)$ for all $t \in I$. Let $\Omega_2(t) := \Omega \setminus \Omega_1(t)$ and assume that $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in I$, then:

$$\bar{\Omega} = \bar{\Omega}_1(t) \cup \bar{\Omega}_2(t), \quad \bar{\Omega}_1(t) \cap \bar{\Omega}_2(t) = \Gamma(t), \quad \partial\Omega_2(t) = \Gamma(t) \cup \partial\Omega.$$

A sketch of the domains is shown in Fig. 2.1.

Remark 2.8. The assumption that the outer boundary is piecewise linear is made to avoid having to analyse perturbation of the domain for Dirichlet boundary conditions, however the presented method and analysis can easily be altered if one removes this assumption.

We label the outer normals of $\Omega_1(t)$ and $\Omega_2(t)$ by $\nu_{\Gamma(t)}$ and $\nu_{\partial\Omega_2(t)}$ respectively. Let:

$$\mathcal{Q}_i := \bigcup_{t \in I} \Omega_i(t) \times \{t\}, \quad \mathcal{Q} := \Omega \times I.$$

Furthermore, we assume there exists a given, global velocity field \mathbf{w} transporting $\Omega_1(t)$ and $\Omega_2(t)$, i.e. $\mathbf{w} \cdot \nu_{\Gamma(t)}|_{\Gamma(t)} = V_\Gamma$ where V_Γ is the normal velocity of $\Gamma(t)$, $\mathbf{w} \cdot \nu_{\partial\Omega_2(t)}|_{\partial\Gamma(t)} = -V_\Gamma$ and $\mathbf{w} \cdot \nu_{\partial\Omega_2(t)}|_{\partial\Omega} = 0$. Throughout the paper this velocity is assumed to be of regularity $\mathbf{w} \in C(I; C(\bar{\Omega}; \mathbb{R}^d))$ with $\mathbf{w}_i(t; \cdot) \in C^2(\bar{\Omega}_i(t); \mathbb{R}^d)$. Let $\Phi_i(t; x) : \bar{\Omega}_i(0) \rightarrow \bar{\Omega}_i(t)$ be the solution to the ordinary differential equations:

$$\begin{aligned} \frac{d}{dt} \Phi_i(t; x) &= \mathbf{w}(t; \Phi_i(t; x)) & x \in \Omega_i(0), \\ \Phi_i(0; x) &= x. \end{aligned} \tag{2.3}$$

We assume the solution exists and is of regularity $\Phi_i \in C^1(\bar{\mathcal{Q}}_i; \mathbb{R}^d)$ with $\Phi_i(t; \cdot) : \bar{\Omega}_i(0) \rightarrow \bar{\Omega}_i(t)$ and $\Phi_i(t; \cdot) \in C^2(\Omega_i(0); \mathbb{R}^d)$, see [46, Thm. 1.45] and [30, Thm. II.1.1, Sec. V] for the necessary additional conditions. Furthermore, both $\Phi_i(t; \cdot)$ are invertible diffeomorphisms for all $t \in I$ with $\text{Im}(\Phi_i(t; \cdot)) = \Omega_i(t)$. We denote by $\Phi_i(-t, \cdot)$ the inverse of $\Phi_i(t, \cdot)$. Since we assumed $\mathbf{w} \in C(I; C(\bar{\Omega}; \mathbb{R}^d))$, it follows that $\Phi_1(t; x)|_{\Gamma(0)} = \Phi_2(t; x)|_{\Gamma(0)}$.

Remark 2.9. For the abstract formulation of the problem, it is only required to assume that the velocity field \mathbf{w} is of sufficient regularity. However, for the purpose of evolving the mesh later, we will assume this velocity field is known explicitly.

Let J_i^t denote the determinant of Jacobian matrix, $J_i^t = \det[\nabla \Phi_i(t; x)]$. The prior assumptions imply $J_i^t(\cdot) \in C^1(\bar{\Omega}_i(0); \mathbb{R})$ and there exists C_Ω independent of time and space such that:

$$\frac{1}{C_\Omega} \leq |J_i^t| \leq C_\Omega,$$

and J_i^{-t} denotes its inverse.

Remark 2.10. Note that we have assumed the global parametric velocity \mathbf{w} is given. The solution of the partial differential equation system is independent of \mathbf{w} apart from the requirements that $\mathbf{w}_\Gamma \cdot \nu_{\Gamma(t)} = V_\Gamma$ and $\mathbf{w} \cdot \nu_{\partial\Omega} = 0$. However the evolving mesh does depend on \mathbf{w} hence the discrete solution depends on the full parametric velocity.

Let $d_\Gamma(t; x)$ be the signed distance function to $\Gamma(t)$:

$$d_\Gamma(t; x) = \begin{cases} -\inf\{|x - y| : y \in \Gamma(t)\}, & \text{for } x \in \bar{\Omega}_1(t), \\ \inf\{|x - y| : y \in \Gamma(t)\}, & \text{for } x \in \Omega_2(t). \end{cases}$$

Then, since the interface is of class C^2 , there exists a constant $\delta > 0$ such that if $x \in \mathcal{N}_{\Gamma(t)} := \{x \in \Omega, |d_\Gamma(t; x)| \leq \delta\}$, it can be uniquely decomposed as:

$$x = d_\Gamma(t; x)\nu_{\Gamma(t)}(\Pi_t(x)) + \Pi_t(x), \quad (2.4)$$

where $\Pi_t(\cdot)$ is the nearest point on $\Gamma(t)$, i.e. $\Pi_t(x) := \inf\{|y - x| : y \in \Gamma(t)\}$ (see [42, Sec. 2.3]). We refer to the set $\mathcal{N}_{\Gamma(t)}$ as the *tubular neighbourhood* of $\Gamma(t)$. Note that δ can be chosen independently of time by the fact that I is compact. Moreover, via the assumption that $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in I$, δ can be chosen small enough such that $\mathcal{N}_{\Gamma(t)} \cap \partial\Omega = \emptyset$ for all $t \in I$.

For the error analysis, we require further regularity of the flow to yield the results collected in the following lemma.

Lemma 2.11. *Let $\Theta \in \mathbb{N}$. Assume further regularity on the flow map $\Phi \in C^{2+\Theta}(I; C^{2+\Theta}(\bar{\Omega}_i(t); \mathbb{R}^d))$ and the initial surface $\Gamma(0)$ is class $C^{2+\Theta}$, then the following geometric quantity have additional regularity:*

- $\Gamma(t)$ is of class $C^{2+\Theta}$,
- $J_i^t(\cdot) \in C^{1+\Theta}(\bar{\Omega}_i(0); \mathbb{R})$,
- $d_\Gamma(t; \cdot) \in C^{2+\Theta}(\mathcal{N}_{\Gamma(t)}; \mathbb{R})$,
- $\Pi_t(\cdot) \in C^{1+\Theta}(\mathcal{N}_{\Gamma(t)}; \mathbb{R}^d)$.

See [24] and [26, Lem. 14.16].

2.3. Realisation. For a given function v acting on Ω , we decompose it as:

$$v_1 := v\chi_{\bar{\Omega}_1(t)}, \quad v_2 := v\chi_{\bar{\Omega}_2(t)},$$

where $\chi_{\bar{\Omega}_i(t)} = 1$ if $x \in \bar{\Omega}_i(t)$ and zero otherwise. A function v on Ω will be identified as the pair $v = (v_1, v_2)$. The jump operator $[[\cdot]]_{\Gamma(t)} : C(\bar{\Omega}_1(t); \mathbb{R}) \times C(\bar{\Omega}_2(t); \mathbb{R}) \rightarrow C(\Gamma(t); \mathbb{R})$ as:

$$[[v]]_{\Gamma(t)} := [v_1 - v_2]|_{\Gamma(t)}.$$

This functional has a natural extension on the Cartesian product of standard Sobolev spaces $H^1(\Omega_1(t)) \times H^1(\Omega_2(t))$ via use of the trace maps; $T_i(t) : H^1(\Omega_i(t)) \rightarrow H^{1/2}(\partial\Omega_i(t))$, $i \in \{1, 2\}$, (see [45, Sec. 7.2.5] for an extensive definition of the trace map) as:

$$[[v]]_{\Gamma(t)} := [T_1(t)v_1 - T_2(t)v_2]|_{\Gamma(t)}.$$

We define the following spaces:

$$\begin{aligned} H(t) &:= L^2(\Omega_1(t)) \times L^2(\Omega_2(t)), \quad \|v\|_{H(t)}^2 := \sum_{i=1}^2 \|v_i\|_{L^2(\Omega_i(t))}^2, \\ V(t) &:= \{v \in H^1(\Omega_1(t)) \times H^1(\Omega_2(t)), \llbracket v \rrbracket_{\Gamma(t)} = 0, v_2|_{\partial\Omega} = 0\}, \quad \|v\|_{V(t)}^2 := \sum_{i=1}^2 \|v_i\|_{H^1(\Omega_i(t))}^2 \\ Z_k(t) &:= \{w \in V(t) \mid w_i \in H^{1+k}(\Omega_i(t))\}, \quad \|w\|_{Z_k(t)}^2 := \sum_{i=1}^2 \|w_i\|_{H^{1+k}(\Omega_i(t))}^2. \end{aligned}$$

Note that due to the continuity of the trace operators $T_i(t)$ on $H^1(\Omega_i(t))$, $V(t)$ defines a closed subspace of $H^1(\Omega_1(t)) \times H^1(\Omega_2(t))$ and contains $H_0^1(\Omega_1(t)) \times H_0^1(\Omega_2(t))$, hence is dense within $H(t)$. For an element $v = (v_1, v_2) \in V(t)$, we will identify $v|_{\Omega_1(t)} = v_1$ and $v|_{\Omega_2(t)} = v_2$. We also define the interface space:

$$H^{1/2}(\Gamma(t)) = \{v \in L^2(\Gamma(t)), |v|_{H^{1/2}(\Gamma(t))} < \infty\}, \quad |v|_{H^{1/2}(\Gamma(t))} := \int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|v(x) - v(y)|^2}{|x - y|^d} dx dy,$$

with norm given by:

$$\|v\|_{H^{1/2}(\Gamma(t))}^2 := \|v\|_{L^2(\Gamma(t))}^2 + |v|_{H^{1/2}(\Gamma(t))}^2.$$

Then, $H^{1/2}(\Gamma(t))$ is a Hilbert space and moreover is dense and compactly embedded in $L^2(\Gamma(t))$ (see [39, Sec. 2]). For consistency of notation, let $\mathcal{V}_\Gamma(t) = H^{1/2}(\Gamma(t))$ and $\mathcal{H}_\Gamma(t) = L^2(\Gamma(t))$, and identify the Hilbert triple $\mathcal{V}_\Gamma(t) \subset \mathcal{H}_\Gamma(t) \subset \mathcal{V}_\Gamma^*(t)$.

Now for a function $v \in H(t)$ and $w \in \mathcal{H}_\Gamma(t)$, the respective flows are defined as:

$$\phi_t v := (v_1(t; \Phi_1(-t; x)), v_2(t; \Phi_2(-t; x))), \quad \phi_t w := w(t; \Phi_1(-t; x)).$$

Lemma 2.12. *The pairs $(V(t), \phi_t)|_{t \in I}$, $(H(t), \phi_t)|_{t \in I}$, $(V^*(t), \phi_{-t}^*)|_{t \in I}$, $(\mathcal{V}_\Gamma(t), \phi_t)|_{t \in I}$, $(\mathcal{H}_\Gamma(t), \phi_t)|_{t \in I}$ and $(\mathcal{V}_\Gamma^*(t), \phi_{-t}^*)|_{t \in I}$ are all compatible.*

Assuming the added regularity $\Phi_i(t; \cdot) \in C^{1+k}(\Omega_i(0); \mathbb{R}^d)$, the pair $(Z_k(t), \phi_t)|_{t \in I}$ is compatible.

Proof. Ass. B1 to B3 need to be checked. This will be checked only for $V(t)$ as a similar logic can be employed for the remaining spaces. B1 follows from both $\Phi_1(-t; x)$, $\Phi_2(-t; x)$ being invertible diffeomorphisms. For B2, via simple manipulation:

$$\begin{aligned} \|\phi_t v\|_{V(t)}^2 &= \sum_{i=1}^2 \int_{\Omega_i(t)} |v_i(t; \Phi_i(-t; x))|^2 + |\nabla v_i(t; \Phi_i(-t; x))|^2, \\ &= \sum_{i=1}^2 \int_{\Omega_i(0)} [|v_i(t; x)|^2 + |[\nabla \Phi_i(-t; y)]^T|_{y=\Phi_i(t; x)} \nabla v_i(t; x)|^2] J_i^t \\ &\leq c(|J_i^t|_{L^\infty(\Omega_i(0))}, |\nabla \Phi_i(-t, x)|_{L^\infty(\Omega_i(t))}) \|v\|_{V(0)}^2, \end{aligned} \tag{2.5}$$

The bound follows from the assumption on the regularity of the velocity field. The same method shows a similar bound for $\|\phi_{-t} \tilde{v}\|_{V(0)} \leq c \|\tilde{v}\|_{V(t)}$ for all $\tilde{v} \in V(t)$. To show measurability, B3, note that the second equality in (2.5) is continuous. For the compatibility of the boundary spaces $\mathcal{V}_\Gamma(t)$, $\mathcal{H}_\Gamma(t)$ and $\mathcal{V}_\Gamma^*(t)$, see [5, Sec. 4 and 5]. Under the added regularity the compatibility of $(Z_k(t), \phi_t)|_{t \in I}$ follows similarly. \square

We will identify both $V(t) \subset H(t) \subset V^*(t)$ and $\mathcal{V}_\Gamma(t) \subset \mathcal{H}_\Gamma(t) \subset \mathcal{V}_\Gamma^*(t)$ with the structure $X(t) \subset Y(t) \subset X^*(t)$ developed in Sec. 2.1.

Lemma 2.13. *The moving space equivalence is satisfied between $W(V, V^*)$ and $\mathcal{W}(V(0), V^*(0))$.*

Proof. The proof follows similarly from the one given for [3, Prop. 7.4] as by assumption the Jacobian determinate is at least of regularity $J_i^t \in C^1(I; C^1(\overline{\Omega}_i(0); \mathbb{R}))$. \square

Remark 2.14. It does not matter which of the flows $\Phi_i(t; x)$ is used to define $\mathcal{H}_\Gamma(t)$ as $\Phi_1(t; x)|_{\Gamma(0)} = \Phi_2(t; x)|_{\Gamma(0)}$. Moreover, it can be shown that $v = (v_1, v_2) \in V(t)$ if, and only if, $v_1 \chi_{\Omega_1(t)} + v_2(1 - \chi_{\Omega_1(t)}) \in H_0^1(\Omega)$ with equivalent norms, hence the space $V(t)$ can be thought as an identification of the components of a functions in $H_0^1(\Omega)$.

We may define both moving space triples $L_V^2 \subset L_H^2 \subset L_{V^*}^2$ and $L_{V_\Gamma}^2 \subset L_{\mathcal{H}_\Gamma}^2 \subset L_{V_\Gamma^*}^2$.

Theorem 2.15 (Reynolds' Transport Theorem). *Let $g_i \in C^1(\mathcal{Q}_i; \mathbb{R})$, then:*

$$\frac{d}{dt} \sum_{i=1}^2 \int_{\Omega_i(t)} g_i = \sum_{i=1}^2 \int_{\Omega_i(t)} \partial_t g_i + \mathbf{w} \cdot \nabla g_i + g_i \nabla \cdot \mathbf{w}.$$

Proof. We use another version of Reynolds' Transport Theorem given in [42, Sec. 2.5]. For $g = (g_1, g_2) \in C^1(\mathcal{Q}_1; \mathbb{R}) \times C^1(\mathcal{Q}_2; \mathbb{R})$, then:

$$\frac{d}{dt} \int_{\Omega \setminus \Gamma(t)} g = \int_{\Omega \setminus \Gamma(t)} \partial_t g - \int_{\Gamma(t)} \llbracket g \rrbracket_{\Gamma(t)} V_\Gamma.$$

Note that here $\Omega \setminus \Gamma(t) = \Omega_1(t) \cup \Omega_2(t)$, and:

$$\begin{aligned} \sum_{i=1}^2 \int_{\Omega_i(t)} \mathbf{w} \cdot \nabla g_i + g_i \nabla \cdot \mathbf{w} &= \sum_{i=1}^2 \int_{\Omega_i(t)} \nabla \cdot (\mathbf{w} g_i) = - \int_{\partial \Omega} [\mathbf{w} g_i] \cdot \nu_{\partial \Omega} - \int_{\Gamma(t)} \llbracket \mathbf{w} g \rrbracket_{\Gamma(t)} \cdot \nu_{\Gamma(t)}, \\ &= - \int_{\Gamma(t)} \llbracket g \rrbracket_{\Gamma(t)} V_\Gamma. \end{aligned} \quad \square$$

Note that via use of the chain rule and the definition of $\Phi(t; \cdot)$ (2.3), for a function $\eta_i \in C^1(\mathcal{Q}_i; \mathbb{R})$:

$$\begin{aligned} \left[\frac{d}{dt} (\eta_i(t; \Phi_i(t; x))) \right] \Big|_{x=\Phi_i(-t; y)} &= \partial_t \eta_i(t; y) + \partial_t (\Phi_i(t; x))|_{x=\Phi_i(-t; y)} \cdot \nabla \eta_i(t; x), \\ &= \partial_t \eta_i(t; y) + \mathbf{w}(t; y) \cdot \nabla \eta_i(t; x). \end{aligned}$$

Giving us back the classical definition for the material derivative, see [27, Sec 1.1.1]. For a function $v \in C_V^1$, we define:

$$\partial_t^\bullet v = \phi_t \frac{d}{dt} (v_1(t; \Phi_1(t; x)), v_2(t; \Phi_2(t; x))) = ([\partial_t + \mathbf{w} \cdot \nabla] v_1(t; x), [\partial_t + \mathbf{w} \cdot \nabla] v_2(t; x)) =: (\partial_t^\bullet v_1, \partial_t^\bullet v_2).$$

One can check that due to the regularity of the flow, the assumptions D1 to D3 are satisfied on the triple $V(t) \subset H(t) \subset V^*(t)$, moreover, via Reynold's transport theorem, one can check that the bilinear form λ introduced Sec. 2.1 in this case becomes:

$$\lambda(t; v, \eta) = (\nabla \cdot \mathbf{w} v, \eta)_{H(t)} = \sum_{i=1}^2 \int_{\Omega_i(t)} [\nabla \cdot \mathbf{w}] v_i \eta_i. \quad (2.6)$$

See [3, Lem. 6.3] for more details.

2.4. The Weak Formulation. Taking the strong problem (1.1), assuming there exists a regular enough solution u , we can rewrite the partial differential equation as:

$$\partial_t^\bullet u_i + \nabla \cdot \mathbf{w} u_i - \nabla \cdot (\mathcal{A}_i(t; x) \nabla u_i) + [\mathcal{B}_i(t; x) - \mathbf{w}] \cdot \nabla u_i + [\mathcal{C}_i(t; x) - \nabla \cdot \mathbf{w}] u_i = f_i. \quad (2.7)$$

Here the term $\nabla \cdot \mathbf{w}$, corresponding to the previously identified bilinear form $\lambda(t; \cdot, \cdot)$, (2.6), is introduced to get the equation in a more convenient form. Then testing with a function $v \in L_V^2$ and using the interface condition, we arrive at the following variational problem:

$$\begin{aligned} \int_0^T \langle \partial_t^\bullet u, v \rangle_{V(t)} dt + \int_0^T \lambda(t; u, v) + \underbrace{\sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{A}_i \nabla u_i \cdot \nabla v_i + [\mathcal{B}_i - \mathbf{w}] \cdot \nabla u_i v_i + [\mathcal{C}_i - \nabla \cdot \mathbf{w}] u_i v_i}_{=: a(t; u, v)} dt \\ = \int_0^T \underbrace{\langle f, v \rangle_{V(t)} + \langle g, v \rangle_{\mathcal{V}_\Gamma(t)}}_{=: l(t; v)} dt. \end{aligned}$$

Note that here the Hilbert triple structure is used for the duality pairings; $\langle f, v \rangle_{V(t)} = (f, v)_{H(t)}$ and $\langle g, v \rangle_{\mathcal{V}_\Gamma(t)} = (g, v)_{\mathcal{H}_\Gamma(t)}$, and we have the initial condition $u(0) = u_0$. This gives us the weak formulation:

$$\int_0^T \langle \partial_t^\bullet u, v \rangle_{V(t)} + a(t; u, v) + \lambda(t; u, v) dt = \int_0^T l(t; v) dt, \quad (2.8)$$

for all $v \in L_V^2$. Moreover if, instead $v \in W(V, V^*)$, we get the equivalent formulation via the transport theorem (for notational convenience later on, we will label the inner product $(\cdot, \cdot)_{H(t)} =: m(t; \cdot, \cdot)$):

$$\frac{d}{dt} m(t; u, v) + a(t; u, v) = m(t; u, \partial_t^\bullet v) + l(t; v).$$

If $\partial_t^\bullet u \in L_H^2$, then via identification of the Hilbert triple, we have:

$$\langle \partial_t^\bullet u, v \rangle_{V(t)} = m(t; \partial_t^\bullet u, v),$$

and the problem can be restated abstractly in this case as $u \in W(V, H)$ being the solution to:

$$m(t; \partial_t^\bullet u, v) + a(t; u, v) + \lambda(t; u, v) = l(t; v), \quad (2.9)$$

for almost all $t \in I$, and all $v \in L_V^2$.

2.5. Well Posedness.

Theorem 2.16. *Assume the following:*

- A1 The coefficients $\mathcal{A}_i \in C(\overline{\mathcal{Q}_i}; \mathbb{R}^{d \times d})$, $\mathcal{B}_i \in C(\overline{\mathcal{Q}_i}; \mathbb{R}^d)$ and $\mathcal{C}_i \in C(\overline{\mathcal{Q}_i}; \mathbb{R})$;
- A2 There exists a constant $\gamma > 0$ such that:

$$\inf_{t \in I} \inf_{x \in \Omega_i(t)} \mathcal{A}_i(t; x) \xi \cdot \xi^T \geq \gamma |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \quad (2.10)$$

- A3 $(u_0, f, g, \mathbf{w}_i, \mathbf{w}) \in H(0) \times L_{V^*}^2 \times L_{V_\Gamma^*}^2 \times C^1(\overline{\mathcal{Q}_i}, \mathbb{R}^d) \times C(I \times \overline{\Omega}; \mathbb{R}^d)$,

then there exists a unique solution $u \in W(V, V^*)$ to (2.8) with inequality:

$$\|u\|_W \leq C \left(\|f\|_{L_{V^*}^2} + \|g\|_{L_{V_\Gamma^*}^2} + \|u_0\|_{H(0)} \right).$$

Furthermore, if it holds that:

- A4 $(u_0, f, g, \mathcal{A}_i) \in V(0) \times L_H^2 \times W(\mathcal{V}_\Gamma, \mathcal{V}_\Gamma^*) \times C^1(\overline{\mathcal{Q}_i}; \mathbb{R}^{d \times d})$, and \mathcal{A}_i is symmetric,

then the solution is of additional regularity $u \in W(V, H)$ with bound:

$$\|u\|_{W(V, H)} \leq C \left(\|f\|_{L_H^2} + \|u_0\|_{V(0)} + \|g\|_{W(\mathcal{V}_T, \mathcal{V}_T^*)} \right).$$

Proof. The existence and uniqueness follows from a standard application of the Babuska-Lax-Milgram theorem in conjunction with Poincaré's inequality, detailed in [4, Thm. 3.6]. The proof of additional regularity under Ass. A4 is given in the Appendix (Lem. A.4). \square

Furthermore, in order to analyse the error in the finite element approximation of the material derivative, it is convenient to define notation for the derivative of the bilinear form $a(t; \cdot, \cdot)$ to be:

$$b(t; v, w) := \frac{d}{dt}[a(t; v, w)] - a(t; \partial_t^\bullet v, w) - a(t; v, \partial_t^\bullet w), \quad \forall v, w \in W(V, V). \quad (2.11)$$

Then, assuming furthermore that $\mathcal{A}_i \in C^1(\overline{\mathcal{Q}}_i; \mathbb{R}^{d \times d})$, $\mathcal{B}_i \in C^1(\overline{\mathcal{Q}}_i; \mathbb{R}^d)$ and $\mathcal{C}_i \in C^1(\overline{\mathcal{Q}}_i; \mathbb{R})$, the bilinear form $b(t; \cdot, \cdot)$ exists and can be explicitly calculated as:

$$\begin{aligned} b(t; v, w) = & \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{D}_i^{\mathcal{A}}(\mathbf{w}, \mathcal{A}_i, v_i, w_i) + \mathcal{D}_i^{\mathcal{B}}(\mathbf{w}, \mathcal{B}_i, v_i, w_i) \\ & + v_i w_i \partial_t^\bullet [\mathcal{C}_i - \nabla \cdot \mathbf{w}] + \nabla \cdot \mathbf{w} [\mathcal{C}_i - \nabla \cdot \mathbf{w}] v_i w_i, \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} \mathcal{D}_i^{\mathcal{A}}(\mathbf{w}, \mathcal{A}_i, v_i, w_i) &= (\partial_t^\bullet \mathcal{A}_i(t; x) + \nabla \cdot \mathbf{w} \mathcal{A}_i(t; x)) \nabla v_i \cdot \nabla w_i - 2D_i(\mathbf{w}, \mathcal{A}_i) \nabla v_i \cdot \nabla w_i, \\ \mathcal{D}_i^{\mathcal{B}}(\mathbf{w}, \mathcal{B}_i, v_i, w_i) &= \partial_t^\bullet [\mathcal{B}_i(t; x) - \mathbf{w}] \cdot \nabla v_i w_i + [\mathcal{B}_i(t; x) - \mathbf{w}] \cdot \nabla v_i w_i \nabla \cdot \mathbf{w} \\ &\quad - \sum_{j,k=1}^d [\mathcal{B}_i - \mathbf{w}]_j (\nabla_j \mathbf{w}_k) \nabla_k v_i w_i, \\ [D_i(\mathbf{w}, \mathcal{A}_i)]_{jl} &= \frac{1}{2} \sum_{r=1}^d [\mathcal{A}_i(t; x)]_{jr} \nabla_r \mathbf{w}_l + [\mathcal{A}_i(t; x)]_{lr} \nabla_r \mathbf{w}_j. \end{aligned}$$

Note that the derivative of the bilinear form $m(t; \cdot, \cdot)$ is already assumed to exist and equals $\lambda(t; \cdot, \cdot)$ introduced in Sec. 2.1.

3. EVOLVING FINITE ELEMENTS

From this point on, we assume the additional geometric regularity as described in Lem. 2.11. We begin by detailing the initial triangulation of the domain and follow with the construction of the evolving mesh. In order to relate discrete and continuous functions we introduce the concept of a *lift* mapping and then finally define evolving finite element spaces.

3.1. Construction of the Initial Domain.

Initial Mesh Construction/Assumption:

- M1 We first perform a partition into d -dimensional simplices corresponding to a polyhedral approximation $\tilde{\Omega}_1^h$ of the interior domain $\overline{\Omega}_1(0)$, $\tilde{\Omega}_1^h = \cup_{j=1}^{M_1} \tilde{K}_1^j$, $\tilde{\mathcal{J}}_1^h := \{\tilde{K}_1^j\}_{j=1}^{M_1}$, where \tilde{K}_1^j are the simplicial elements of positive diameter, bounded by some \tilde{h} , and M_1 is the number of elements.

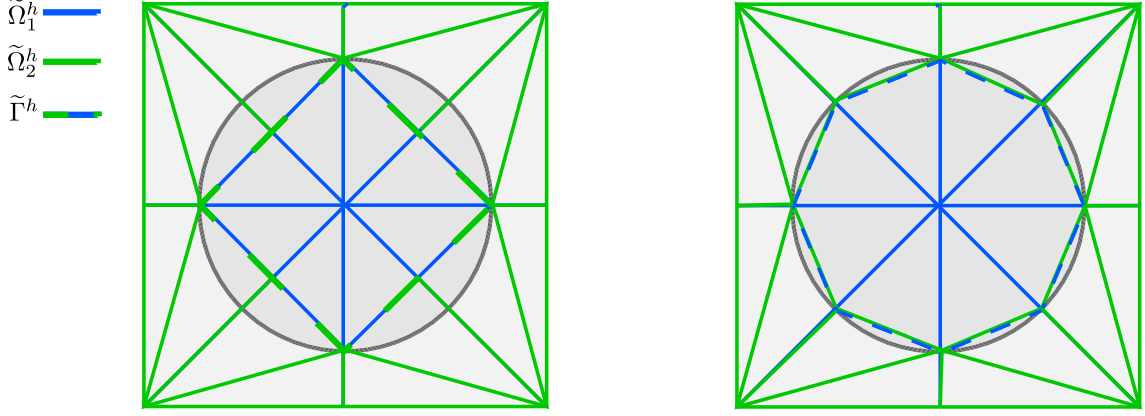


FIGURE 3.1. Showing the difference between a non viable initial mesh and an adequate one for a circle enclosed in a square. The one on the left breaking condition [M5](#) whereas the one on the right following condition [M5](#).

- M2 The set $\tilde{\Omega}_2^h := \overline{\Omega \setminus \tilde{\Omega}_1^h}$ is polyhedral and we construct a partition into d -dimensional simplices $\tilde{\mathcal{J}}_1^h := \{\tilde{K}_2^j\}_{j=1}^{M_2}$ with maximum diameter \tilde{h} . Let $\tilde{\mathcal{J}}^h = \cup_{i=1}^2 \tilde{\mathcal{J}}_i^h$ assume that all partitions $\{\tilde{\mathcal{J}}_1^h, \tilde{\mathcal{J}}_2^h, \tilde{\mathcal{J}}^h\}$ are admissible, shape regular and quasi-uniform in $\{\tilde{\Omega}_1^h, \tilde{\Omega}_2^h, \tilde{\Omega}\}$ respectively, see [\[12, Def. 5.1\]](#).
- M3 Each element \tilde{K} contains $d+1$ facets labelled $\{\tilde{E}^j\}_{j=1}^{d+1} \subset \tilde{K}$. We refer to the set of all facets of all elements in $\tilde{\mathcal{J}}^h$ by $\tilde{\mathcal{J}}_\partial^h$.
- M4 For $\tilde{E} \in \tilde{\mathcal{J}}_\partial^h$, if there exists $\tilde{K}_1 \in \tilde{\mathcal{J}}_1^h$ and $\tilde{K}_2 \in \tilde{\mathcal{J}}_2^h$ such that $\tilde{E} = \tilde{K}_1 \cap \tilde{K}_2$, then we call \tilde{E} an *interface facet* and label the collection of those facets $\tilde{\mathcal{J}}_\Gamma^h$ and the union of interface facets $\tilde{\Gamma}^h$. If for a given \tilde{E} , there is only one element $\tilde{K} \in \tilde{\mathcal{J}}_2^h$ such that $\tilde{E} \subset \tilde{K}$, then such a facet is called a *boundary facet*.
- M5 We restrict the vertices of interface facets to be on $\Gamma(0)$, i.e, if \tilde{E} is an interface facet, and $\{\tilde{a}_E^j\}_{j=1}^{d-1}$ are the vertices of \tilde{E} , then $\{\tilde{a}_E^j\}_{j=1}^{d-1} \subset \Gamma(0)$. Conversely, we will assume that if a facet has all its vertices on the interface, then it is an interface facet. See [Fig. 3.1](#) for an example.
- M6 Let \hat{K}_{ref} be the reference element of $\tilde{\mathcal{J}}$ (i.e for all $\tilde{K} \in \tilde{\mathcal{J}}$, there exists an invertible affine map $F_{\tilde{K}}$ such that $F_{\tilde{K}}(\hat{K}_{\text{ref}}) = \tilde{K}$). The reference element is then equipped with the standard k^{th} ($k \in (0, \Theta)$) Lagrangian element triple $(\hat{K}_{\text{ref}}, \hat{P}^k, \hat{\Sigma}^k)$ (see [\[13, Sec. 3.2\]](#)) where \hat{P}^k is the set of k^{th} order Lagrange polynomials and $\hat{\Sigma}^k$ is the dual basis of \hat{P}^k , which in this case takes the form $\hat{\Sigma} = \{\chi \rightarrow \chi(\hat{\alpha}), \hat{\alpha} \in N(\hat{K}_{\text{ref}})\}$, where $N(\hat{K}_{\text{ref}})$ is the set of Lagrangian nodes in \hat{K}_{ref} . Let $(\tilde{K}, \tilde{P}^k, \tilde{\Sigma}^k)$ and $(\tilde{K}', \tilde{P}'^k, \tilde{\Sigma}'^k)$ be two adjacent elements in $\tilde{\mathcal{J}}$, the following assumption is made

$$\left(\bigcup_{\alpha \in N(\tilde{K})} \tilde{\alpha} \right) \cap \tilde{K}' = \left(\bigcup_{\alpha' \in N(\tilde{K}')} \tilde{\alpha}' \right) \cap \tilde{K},$$

i.e the Lagrangian nodes are shared between two adjacent elements.

Note that via construction, $\tilde{E} \in \tilde{\mathcal{J}}_\Gamma^h$, if and only if there exists an element $\tilde{K}_1 \in \tilde{\mathcal{J}}_1^h$ and $\tilde{K}_2 \in \tilde{\mathcal{J}}_2^h$ with $\tilde{E} = \tilde{K}_1 \cap \tilde{K}_2$ and hence $\tilde{\Gamma}_0^h = \tilde{\Omega}_1^h(0) \cap \tilde{\Omega}_2^h(0)$. This construction defines Lagrangian triangulated bulk domains $(\tilde{\Omega}_1^h, \tilde{\Omega}_2^h, \tilde{\Omega}^h)$, and $\tilde{\Gamma}_0^h$ defines a triangulated hypersurface, see [\[22, Def. 4.14 and 6.14\]](#). After the initial triangulation, we define the isoparametric version using the same method as [\[22, Sec. 8.5\]](#)

Θ	Degree of additional geometric regularity assumed in Lem. 2.11 .
$(\widehat{K}_{\text{ref}}, \widehat{P}^k, \widehat{\Sigma}^k)$	Standard $k - th$ order Lagrangian reference element, with $k \in (0, \Theta)$.
$\widehat{\alpha}$	Lagrange node of the reference element.
$i \in \{1, 2\}$	As a subscript, will always only refer to which of the domains the quantity appertains.
$\widetilde{\Omega}_i^h, \widetilde{\Gamma}^h$	Initial Triangulation of the domains and interface(at $t = 0$).
$\widetilde{\mathcal{J}}_i^h, \widetilde{\mathcal{J}}^h, \widetilde{\mathcal{J}}_\Gamma^h$	Partitions of $\widetilde{\Omega}_i^h, \overline{\Omega}$ and $\widetilde{\Gamma}^h$ respectively.
$\widetilde{K}, \widetilde{E}, \widetilde{\alpha}, \widetilde{a}$	Element/Facet/Lagrangian node/vertex appertaining to $\widetilde{\mathcal{J}}^h$.
Ψ^h	Diffeomorphism map $\Psi^h: \widetilde{\Omega}_i^h \rightarrow \overline{\Omega}_i$.
$\Pi_0(x)$	Minimal distance projection onto $\Gamma(0)$.
$\Omega_i^h(0), \Gamma^h(0)$	Triangulated bulk domains (hypersurface) approximating $\overline{\Omega}_i(0), \Gamma(0)$.
$\mathcal{J}_i^h, \mathcal{J}^h, \mathcal{J}_\Gamma^h$	Partition of isoparametric element of $\Omega_i^h(0), \overline{\Omega}, \Gamma^h(0)$.

TABLE 3.1. List of symbols

which we detail in the following. Let \widehat{K}_{ref} be the reference element of the partition $\widetilde{\mathcal{J}}^h$, with reference map $F_{\widehat{K}}: \widehat{K}_{\text{ref}} \rightarrow \widehat{K}$. For $\eta \in C(\overline{\Omega}; \mathbb{R})$ and for some $\widehat{K} \in \widetilde{\mathcal{J}}^h$, we define the interpolation operator element-wise:

$$\widetilde{I}^h(\eta)|_{\widehat{K}} := \sum_{\{j: \chi^j \in \widetilde{P}^k\}} \sigma^j(\eta) \chi^j = \sum_{\{j: \chi^j \in \widetilde{P}^k\}} \eta(\widetilde{\alpha}_{\widehat{K}}^j) \chi^j.$$

Let $\{\widetilde{a}_{\widehat{K}}^j\}_{j=1}^{d+1}$ be the vertices of an element $\widehat{K} \in \widetilde{\mathcal{J}}^h$. If two or more of the vertices are on the interface $\Gamma(0)$, then the element is referred to as an *interface element*. Let $\widetilde{\mathcal{F}}$ be the set of all interface elements and define the following function $\Psi^h: \Omega \rightarrow \Omega$ element-wise as follows. If $\widehat{K} \notin \widetilde{\mathcal{F}}$, then $\Psi^h(x) = x$ for $x \in \widehat{K}$. If instead $\widehat{K} \in \widetilde{\mathcal{F}}$, then expand $x \in \widehat{K}$ into barycentric coordinates:

$$x = \sum_{j=1}^{d+1} \mu_j(x) \widetilde{a}_{\widehat{K}}^j.$$

Let L_K be the number of vertices in \widehat{K} that lie on $\Gamma(0)$ ($L_K \geq 2$ by assumptions) and assume that the vertices are ordered so that the first L_K lie on $\Gamma(0)$. Let:

$$\widetilde{\mu}_K(x) := \sum_{j=1}^{L_K} \mu_j(x), \quad \sigma_{\widehat{K}} := \{x \in \widehat{K}, \widetilde{\mu}_K(x) = 0\}.$$

From the properties of barycentric coordinates, $\widetilde{\mu}_K$ can be seen as the distance from the discrete interface, with $\widetilde{\mu}_K(x) = 1$ when x is on a facet between vertices on the interface, and $\widetilde{\mu}_K(x) = 0$ when x is on the facet spanned by non-interface vertices.

Let

$$y(x) = \sum_{j=1}^{L_K} \frac{\mu_j(x)}{\widetilde{\mu}_K(x)} \widetilde{a}_{\widehat{K}}^j. \tag{3.1}$$

Note that $y(x) \in \tilde{K}$ since $0 \leq \mu_j(x) \leq \tilde{\mu}_K(x)$. Hence define:

$$\Psi^h|_{\tilde{K}}(x) := \begin{cases} x + (\tilde{\mu}_K(x))^{k+2}(\Pi_0(y(x)) - y(x)) & \text{if } x \notin \sigma_{\tilde{K}}, \\ x & \text{otherwise.} \end{cases} \quad (3.2)$$

Where Π_0 is the nearest point projection on $\Gamma(0)$, introduced in (2.4). We summarise the properties of this map in the following theorem. For the definition of *triangulated bulk domain* and *k-bulk finite element*, see [22, Def. 4.14 and 4.5]. We denote by \tilde{I}^h interpolation into the space of polynomials of degree k over \tilde{K} .

Theorem 3.1 ([22], Lem. 4.8 and 8.8). *For \tilde{h} small enough, the map $\Psi^h|_{\tilde{K}} \in C^{k+1}(\tilde{K}; \mathbb{R}^d)$ and is invertible for each $\tilde{K} \in \tilde{\mathcal{J}}^h$ and $\Psi^h : \tilde{\Gamma}_0^h \rightarrow \Gamma(0)$. Define the following:*

$$\begin{aligned} F_K &:= [\tilde{I}^h \Psi^h](F_{\tilde{K}}), \\ K &:= F_K(\hat{K}_{\text{ref}}), \\ P^k &:= \{\hat{\chi}_k \circ F_K^{-1} : \hat{\chi}_k \in \hat{P}^k\}, \\ \Sigma^k &:= \{\chi \mapsto \hat{\sigma}(\chi \circ F_K) : \hat{\sigma} \in \hat{\Sigma}^k\}, \end{aligned}$$

then the triplet (K, P^k, Σ^k) with reference map F_K defines a *k-bulk finite element triplet* ([22, Def. 4.5]). Let $\mathcal{J}_i^h = \{[\tilde{I}^h \Psi^h](\tilde{K}_i), \tilde{K}_i \in \tilde{\mathcal{J}}_i^h\}$, $\mathcal{J}_\Gamma^h = \{[\tilde{I}^h \Psi^h]_1(E), E \in \tilde{\mathcal{J}}_\Gamma^h\}$ (here $[\tilde{I}^h \Psi^h]_1$ refers to taking the interpolation with the adjacent element in $\tilde{\mathcal{J}}_1$), then $\{\mathcal{J}_1^h, \mathcal{J}_2^h, \mathcal{J}_\Gamma^h\}$ are conforming admissible sub-divisions. Furthermore, let:

$$\Omega_i^h(0) := \bigcup_{K_i \in \mathcal{J}_i^h} K_i, \Omega^h := \bigcup_{K \in \mathcal{J}^h} K, \Gamma^h(0) := \bigcup_{E \in \mathcal{J}_\Gamma^h} E,$$

then $(\Omega_1^h(0), \Omega_2^h(0))$ define triangulated bulk domains approximating $(\bar{\Omega}_1(0), \bar{\Omega}_2(0))$, $\Gamma^h(0)$ a triangulated hypersurface approximating $\Gamma(0)$.

Now since we are dealing with an interface problem, we require additionally to check if $\mathcal{J}^h = \mathcal{J}_1^h \cup \mathcal{J}_2^h$ forms a conforming admissible sub-division of the whole domain $\bar{\Omega}$.

Lemma 3.2. *\mathcal{J}^h forms a conforming admissible sub-division of the whole domain $\bar{\Omega}$, moreover, interface facets are mapped to their isoparametric equivalent in such a way that:*

$$\Gamma^h(0) = \bigcup_{E \in \mathcal{J}_\Gamma^h} E = \Omega_1^h \cap \Omega_2^h.$$

Proof. Since \mathcal{J}^h is the union of two admissible conforming subdivision, it only remains to check that if we are given two elements $K_i \in \mathcal{J}_i^h$, then $K_1^\circ \cap K_2^\circ = \emptyset$. It suffices to show the invertibility of the map $\tilde{I}^h \Psi^h$ on $\bar{\Omega}$. For an interface facet \tilde{E} with two adjacent element \tilde{K}_i we require continuity across E : $\Psi^h|_{\tilde{K}_1}(\tilde{E}) = \Psi^h|_{\tilde{K}_2}(\tilde{E})$. By construction of the mesh, $x \in \tilde{E}$, $\mu_{K_1}(x) = \mu_{K_2}(x) = 1$. This implies $y(x) = x$ in (3.1) and hence both maps $\Psi^h|_{\tilde{K}_i}(x) = \Pi_0(x)$ from (3.2). Any Lagrangian node $\tilde{\alpha}_i$ on \tilde{E} will be mapped by both maps to $\alpha_i := \Pi_0(\tilde{\alpha}_i)$. Since each interface facet contains the exact amount of nodes to uniquely define a polynomial on the facet, which must equal the restriction on the interface element of the Lagrangian polynomial on the full element (see [12, Rem. 5.4]), hence for $x \in \tilde{E}$:

$$\begin{aligned} [\tilde{I}^h \Psi^h]|_{\tilde{K}_1}(x) &= \sum_{\{j: \chi^j \in P^k\}} \Psi^h(\tilde{\alpha}_{\tilde{K}_1}^j) \chi^j(x) = \sum_{\{j: \chi^j \in P^k\}} \Pi_0(\tilde{\alpha}_{\tilde{K}_1}^j) \chi^j(x), \\ &= \sum_{\{j: \chi^j \in P^k\}} \Pi_0(\tilde{\alpha}_{\tilde{K}_2}^j) \chi^j(x) = \sum_{\{j: \chi^j \in P^k\}} \Psi^h(\tilde{\alpha}_{\tilde{K}_2}^j) \chi^j(x) = [\tilde{I}^h \Psi^h]|_{\tilde{K}_2}(x). \end{aligned}$$

For $\tilde{E} \in \tilde{\mathcal{J}}_\Gamma^h$, let $\tilde{K}_i \in \tilde{\mathcal{J}}_i^h$ be the adjacent elements to \tilde{E} and $K_i = I^h \Psi^h(\tilde{K}_i)$. Since the map $I^h \Psi^h$ is invertible onto its image for each $\tilde{K} \in \tilde{\mathcal{J}}^h$ and is continuous across the intersection \tilde{E} , it holds that $I^h \Psi^h$ is invertible on $\tilde{K}_1 \cup \tilde{K}_2$, since both elements are closed, hence:

$$E := [I^h \Psi^h]_{\tilde{K}_1}(\tilde{E}) = I^h \Psi^h(\tilde{K}_1 \cap \tilde{K}_2) = I^h \Psi^h(\tilde{K}_1) \cap I^h \Psi^h(\tilde{K}_2) = K_1 \cap K_2.$$

Therefore the image of an interface facet remains an interface facet. Moreover, this shows that $K_1^\circ \cap K_2^\circ = \emptyset$ for any $K_i \in \mathcal{J}_i^h$ and hence \mathcal{J}^h is a conforming admissible sub-division. \square

Fig. 3.2 shows how the map $\tilde{I}^h \Psi^h$ deforms the original mesh. We are initially given two tetrahedral elements of the initial meshes, one in $\tilde{\Omega}_1^h$ and one in $\tilde{\Omega}_2^h$, intersecting on an interface element. Applying the map $\tilde{I}^h \Psi^h$ to this yields isoparametric elements whose intersection is the image of the interface element under $\tilde{I}^h \Psi^h$.

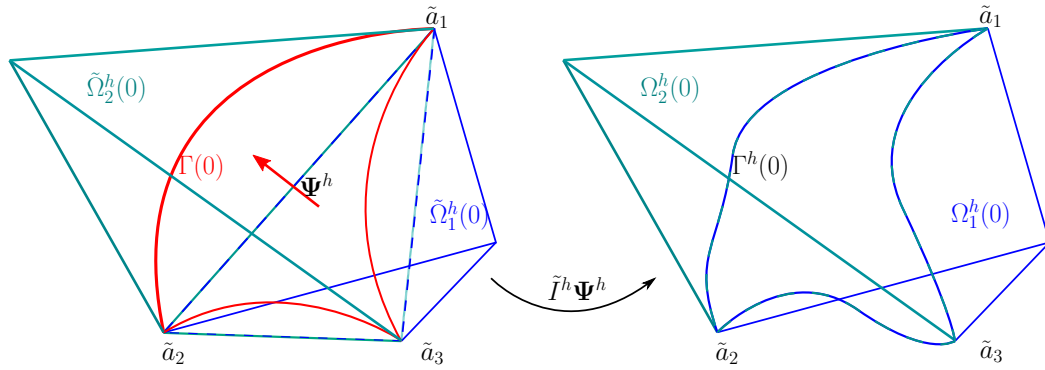


FIGURE 3.2. The intersection of two interface elements (teal and blue respectively) of different domains. The shared interface facet is then pushed by the map Ψ^h to become a piece of $\Gamma(0)$. The map $\tilde{I}^h \Psi^h$ maps the original mesh to an isoparametric mesh approximating the interface.

Let $\alpha_K^j|_{j=1}^{N(k)}$ be the Lagrangian nodes on an element $K \in \mathcal{J}^h$, which by construction are defined as $\alpha_K^j = F_K(\hat{\alpha}_{\text{ref}}^j)$, the corresponding interpolation operator for a given function $\eta \in C(\bar{\Omega}; \mathbb{R})$ is given by:

$$[I^h \eta]_K = \sum_{\{j: \chi^j \in P^k\}} \eta(\alpha_K^j) \chi^j.$$

3.2. Time Dependent Mesh. Define the flow:

$$\Phi_i^h(t; \cdot)|_{K_i(0)} := I_{K_i(0)}^h[\Phi_i(t; \Psi^h \circ (\tilde{I}^h \Psi^h)^{-1}(\cdot))]. \quad (3.3)$$

We denote by $\Phi_i^h(-t, \cdot)|_{K_i(t)}: K_i(t) \rightarrow K_i(0)$ the space-only inverse of $\Phi_i^h(t; \cdot)|_{K_i(0)}$.

Remark 3.3. The flow is defined this way such that it evolves the parametric meshes $\Omega_i^h(0)$. Indeed, decomposing $\Phi_i^h(t; \cdot)$ into its components:

$$(\tilde{I}^h \Psi^h)^{-1}: \Omega_i^h(0) \rightarrow \tilde{\Omega}_i^h, \quad \Psi^h: \tilde{\Omega}_i^h \rightarrow \bar{\Omega}_i(0), \quad \Phi_i(t; \cdot): \bar{\Omega}_i(0) \rightarrow \bar{\Omega}_i(t).$$

Hence, the flow $\Phi_i^h(t; \cdot)$ is a polynomial function approximating the evolution of the domains. Moreover, this does define a proper flow map, as, at $t = 0$:

$$\begin{aligned}\Phi_i^h(0; x)|_{K_i(0)} &= I_{K_i(0)}^h[\Psi^h \circ (\tilde{I}^h \Psi^h)^{-1}(x)] = \sum_{\chi^j \in P^k} \Psi^h \circ [(\tilde{I}^h \Psi^h)^{-1}(\alpha_{K_i}^j)] \chi^j(x), \\ &= \sum_{\chi^j \in P^k} \Psi^h(\tilde{\alpha}_{K_i}^j) \chi^j(x) = \sum_{\chi^j \in P^k} \alpha_{K_i}^j \chi^j(x) = x.\end{aligned}$$

The composition property for $\Phi_i^h(t + s; \cdot) = \Phi_i^h(t; \Phi_i^h(s; \cdot))$ will be shown following the next lemma.

For \tilde{h} small enough, this map is an invertible diffeomorphism on each element $K_0 \in \mathcal{J}^h$. As before, we summarise the construction in the following lemma:

Lemma 3.4. *For \tilde{h} small enough, map $\Phi_i^h(t; \cdot)|_{\tilde{K}} \in C^{k+1}(\tilde{K}; \mathbb{R}^d)$ and is invertible onto its image for each $\tilde{K} \in \tilde{\mathcal{J}}^h$. Moreover, define the following:*

$$\begin{aligned}F_{K(t)}(\cdot) &:= \Phi_i^h(t; F_{\tilde{K}}(\cdot)), \\ K(t) &:= F_{K(t)}(\tilde{K}), \\ P^k(t) &:= \{\chi_k \circ F_{K(t)}^{-1} : \chi_k \in \hat{P}^k\}, \\ \Sigma^k(t) &:= \{\chi \mapsto \sigma(\chi \circ F_{K(t)}) : \sigma \in \hat{\Sigma}^k\},\end{aligned}$$

then the triplet $(K(t), P^k(t), \Sigma^k(t))$ with reference map $F_{K(t)}$ defines a bulk evolving finite element triplet. Let $\mathcal{J}_i^h(t) = \{\Phi_i^h(t; \tilde{K}_i), \tilde{K}_i \in \mathcal{J}_i^h\}$, $\mathcal{J}_\Gamma^h(t) = \{\Phi_1^h(t; E), E \in \mathcal{J}_\Gamma^h\}$ and $\mathcal{J}^h(t) = \mathcal{J}_1^h(t) \cup \mathcal{J}_2^h(t)$, then $\{\mathcal{J}_1^h(t), \mathcal{J}_2^h(t), \mathcal{J}^h(t)\}$ are evolving conforming admissible sub-divisions (see [22, Def. 4.32]). Furthermore, let:

$$\Omega_i^h(t) = \bigcup_{K_i(t) \in \mathcal{J}_i^h(t)} K_i(t), \Omega^h(t) = \bigcup_{K \in \mathcal{J}^h} K(t), \Gamma^h(t) = \bigcup_{E \in \mathcal{J}_\Gamma^h} E = \Omega_1^h(t) \cap \Omega_2^h(t),$$

then $(\Omega_1^h(t), \Omega_2^h(t))$ define triangulated bulk domains approximating $(\bar{\Omega}_1(t), \bar{\Omega}_2(t))$, $\Gamma^h(t)$ is a triangulated hypersurface approximating $\Gamma^h(t)$, and $\Omega^h(t)$ defines a triangulated bulk domain that is an exact partition of $\bar{\Omega}$

Proof. The proof follows the same way as [Lem. 3.2](#). □

For each $K(t) \in \mathcal{J}^h(t)$, let $h_{K(t)}$ be the diameter of the flat simplex whose vertices match $K(t)$. We define $h := \max_{t \in I} \max_{K(t) \in \mathcal{J}^h(t)} \text{diam}(K(t))$ to be the maximum mesh diameter, where $\text{diam}(K(t))$ is the diameter of the affine element whose vertices match $K(t)$ (see [22, Lem. 4.9]).

Remark 3.5. This allows us to move the Lagrangian nodes via $\alpha_{K_i(t)}^j = \Phi_i^h(t; \alpha_{K_i}^j)$. The Lagrangian interpolation operator, $I^h|_{K_i(t)}$, is then defined in the canonical way. Moreover for $x \in K_i(0)$:

$$\begin{aligned}\partial_t \Phi_i^h(t; x) &= \sum_{\{j: \tilde{\chi}^j \in \tilde{P}^k\}} \partial_t \Phi_i(t; \Psi^h(\alpha_{\tilde{K}_i}^j)) \tilde{\chi}^j(x) = \sum_{\{j: \tilde{\chi}^j \in \tilde{P}^k\}} \mathbf{w}(t; \Phi_i \circ \Psi^h(\alpha_{\tilde{K}_i}^j)) \tilde{\chi}^j(x) \\ &= \sum_{\{j: \chi^j \in P^k(t)\}} \mathbf{w}(t; \alpha_{K_i(t)}^j) \chi^j(t; \Phi_i^h(t; x)) =: \mathbf{w}^h(t; \Phi_i^h(t; x)),\end{aligned}$$

where one sees that \mathbf{w}^h is the interpolated velocity with respect to the moving nodes:

$$\mathbf{w}^h(t; \cdot)|_{K_i(t)} = I^h|_{K_i(t)}[\mathbf{w}(t; \cdot)]. \quad (3.4)$$

Hence, element-wise, the discrete flow satisfies ODE:

$$\begin{aligned} \frac{d}{dt} \Phi_i^h(t; x) &= \mathbf{w}^h(t; \Phi_i^h(t; x)), \quad x \in K_i(0), \\ \Phi_i^h(0; x) &= x, \end{aligned} \quad (3.5)$$

and therefore satisfies the composition property $\Phi_i^h(t + s; \cdot) = \Phi_i^h(t; \Phi_i^h(s; \cdot))$, see [30].

It will be assumed that the mesh remains *uniformly quasi-uniform* in time, see [22, Def. 4.35] as the discrete flow Φ^h can deform the mesh significantly. An example of the temporal deformation of an evolving element is shown in Fig. 3.3. Despite interior elements of the initial partition being linear, since the velocity used to displace the elements is a polynomial interpolant of the velocity, the resulting element might not remain linear and can be deformed. An alternative construction, for which interior elements remain affine, is given in [36].

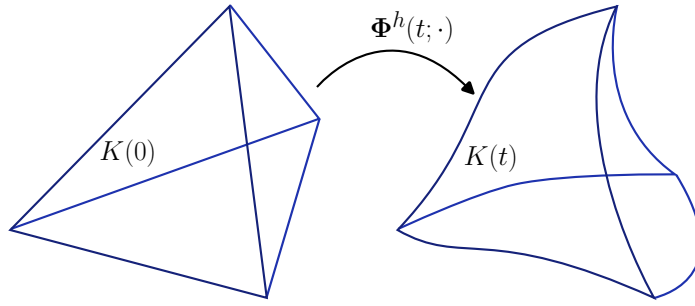


FIGURE 3.3. Example of the temporal deformation of an interior element in three space dimensions.

The *Broken Sobolev* space is defined as follows:

$$\begin{aligned} W_T^{1,p}(\mathcal{J}_i^h(t)) &= \{\eta \in L^1(\Omega_i^h(t)), \eta|_{K_i(t)} \in W^{1,p}(K_i(t)) \quad \forall K_i(t) \in \mathcal{J}_i^h(t), \\ &\quad \eta|_{\partial K_i(t)} = \eta|_{\partial K_i'(t)} \quad \forall K_i'(t) \in \mathcal{J}_i^h(t) \text{ s.t. } K_i'(t) \cap K_i(t) \neq \emptyset\}, \end{aligned}$$

equipped with the *Broken Sobolev* norm:

$$\|\eta^h\|_{W^{1,p}(\mathcal{J}_i^h(t))}^p := \sum_{K_i(t) \in \mathcal{J}_i^h(t)} \|\eta^h\|_{W^{1,p}(K_i(t))}^p, \quad \|\eta^h\|_{W^{1,\infty}(\mathcal{J}_i^h(t))} := \max_{K(t) \in \mathcal{J}^h} \|\eta\|_{W^{1,\infty}(K(t))}.$$

Remark 3.6. The space $W_T^{1,p}(\mathcal{J}_i^h(t))$ is indeed a Banach space see [22, Lem. 4.19].

The discrete spaces are then defined as:

$$\begin{aligned} H^h(t) &:= L^2(\Omega_1^h(t)) \times L^2(\Omega_2^h(t)), \\ V^h(t) &:= \{\eta^h \in W_T^{1,2}(\mathcal{J}_1^h(t)) \times W_T^{1,2}(\mathcal{J}_2^h(t)), \eta_1^h - \eta_2^h|_{\Gamma^h(t)} = 0, \text{ and } \eta_2^h|_{\partial\Omega^h} = 0\}, \end{aligned}$$

equipped with the norms:

$$\|\cdot\|_{V^h(t)}^2 := \sum_{i=1}^2 \|\cdot\|_{W^{1,2}(\mathcal{J}_i^h(t))}^2, \quad \|\cdot\|_{H^h(t)}^2 := \sum_{i=1}^2 \|\cdot\|_{L^2(\Omega_i^h(t))}^2.$$

Define the map $\phi_t^h : H^h(0) \rightarrow H^h(t)$ element-wise as:

$$(\phi_t^h v^h)(x_1, x_2) := (v_1^h(\Phi_1^h(-t; x_1)), v_2^h(\Phi_2^h(-t; x_2))) \quad \text{for } (x_1, x_2) \in K_1(t) \times K_2(t) \subset \Omega_1^h(t) \times \Omega_2^h(t).$$

That is

$$(\phi_t^h v^h)(x_1, x_2) := (v_1^h(y_1), v_2^h(y_2)) \quad \text{for } y_i \in K_i(0), y_i = \Phi_i^h(-t; x_i), x_i \in K_i(t).$$

Lemma 3.7. $(V^h(t), \phi_t^h)_{t \in I}$ and $(H^h(t), \phi_t^h)_{t \in I}$, are compatible pairs.

Proof. This follows by the regularity of the map Φ^h and $\mathcal{J}^h(t)$, see [22, Lem. 4.36]. \square

Hence the moving spaces $L_{V^h}^2$ and $L_{H^h}^2$ are well defined. Denote the discrete material derivative by:

$$\partial_t^h \eta := \phi_t^h \frac{\partial}{\partial t} \phi_{-t}^h \eta, \quad (3.6)$$

for $\eta \in C_{H^h}^1$. The bilinear form $\lambda^h(t; \cdot, \cdot)$ of Def. 2.3 associated with this material derivative is:

$$\lambda^h(t; \eta^h, v^h) = (\nabla \cdot \mathbf{w}^h \eta^h, v^h)_{H^h(t)},$$

where \mathbf{w}^h is the previously defined discrete velocity from Sec. 3.2 (see [22, Lem. 8.10] for derivation). This allows us to define, just as before, the discrete space:

$$W(V^h, V^h) = \{v^h \in L_{V^h}^2, \partial_t^h v^h \in L_{V^h}^2\}.$$

3.3. The lift. The last mesh related concept needed is the *lift map* (see [22, Sec. 8.6]). Fix $t \in I$, if $K(t) \in \mathcal{J}^h(t)$ is an interior element, then define the lift $\Lambda^h(t; \cdot)$ as:

$$\Lambda^h(t; x) = x, \text{ for } x \in K(t).$$

If instead $K(t)$ is an interface element, we first pull-back the reference map to $\hat{x} \in \hat{K}_{\text{ref}}$ such that $x = F_{K(t)}(\hat{x})$, then decomposing \hat{x} into barycentric coordinates with respect to the vertices $\hat{a}_{K_{\text{ref}}}^j$ of \hat{K}_{ref} , we have:

$$\hat{x} = \sum_{j=1}^{d+1} \mu_j(\hat{x}) \hat{a}_{K_{\text{ref}}}^j,$$

and once again, let L_K be the number of vertices on the interface and assume the vertices are ordered so that $\hat{a}_{K_{\text{ref}}}^j$, $j = 1, \dots, L_K$, get mapped on to $\Gamma(t)$, then we introduce the interface distance and the singular set analogously:

$$\tilde{\mu}(\hat{x}) = \sum_{j=1}^{L_K} \mu_j(\hat{x}), \quad \sigma = \{\hat{x} \in \hat{K}_{\text{ref}} \mid \tilde{\mu}(\hat{x}) = 0\}.$$

The projection is now defined on the reference element:

$$\hat{y}(\hat{x}) = \sum_{j=1}^{L_K} \frac{\mu_j(\hat{x})}{\tilde{\mu}(\hat{x})}, \quad y(t; x) := F_{K(t)}(\hat{y}(\hat{x})).$$

Hence the lift operator can now be defined on interface elements as:

$$\Lambda^h(t; x)|_{K(t)} = \begin{cases} x + (\tilde{\mu}(\hat{x}))^{k+2} (\Pi_t(y(t; x)) - y(t; x)) & \text{if } \hat{x} \notin \sigma, \\ x & \text{otherwise.} \end{cases}$$

Then, computing component wise, we see:

$$\partial_t^h \Lambda^h(t; x)|_{K(t)} = \begin{cases} \mathbf{w}^h(t; x) + \partial_t^h [(\tilde{\mu}(\hat{x}))^{k+2} (\Pi_t(y(t; x)) - y(t; x))] & \text{if } \hat{x} \notin \sigma, \\ \mathbf{w}^h(t; x) & \text{otherwise.} \end{cases}$$

Let $z = \Phi_i^h(-t; x)$ (depending on whether $x \in \Omega_i^h(t)$, we also use the shorthand $\Phi_{-t}(x) = \Phi_i^h(-t; x)$ and $\Phi_t(z) = \Phi_i^h(t; z)$, depending on whether $z \in \Omega_i^h(0)$):

$$\begin{aligned} \partial_t^h[(\tilde{\mu}(\hat{x}))^{k+2}(\Pi_t(y(t; x)) - y(t; x))] &= \phi_t^h \frac{d}{dt}[(\tilde{\mu}(F_{\hat{K}}^{-1}z))^{k+2}(\Pi_t(y(t; \Phi_t^h(z))) - y(t; \Phi_t^h(z))), \\ &= \tilde{\mu}(\hat{x})^{k+2} \phi_t^h \frac{d}{dt}[\Pi_t(y(t; \Phi_t^h(z))) - y(t; \Phi_t^h(z))], \\ &= \tilde{\mu}(\hat{x})^{k+2} [\partial_t \Pi_t(y(t; x)) + \partial_t^h y(t; x) \cdot [\nabla \Pi_t(y(t; x))] - \partial_t^h y(t; x)]. \end{aligned}$$

The formula for $\partial_t^h y(t; x)$ can also be explicitly found:

$$\partial_t^h y(t; x) = \phi_t^h \frac{d}{dt} F_{K(t)}(\tilde{y}(t; F_{\hat{K}}^{-1}z)) = \phi_t^h \mathbf{w}^h(t; F_{K(t)}\tilde{y}(t; F_{\hat{K}}^{-1}z)) = \mathbf{w}^h(t; y(t; x)),$$

by use of (3.3) and the definition of $F_{K(t)}$. Hence:

$$\begin{aligned} \partial_t^h \Lambda^h(t; x)|_{K(t)} & \tag{3.7} \\ &= \begin{cases} \mathbf{w}^h(t; x) + \tilde{\mu}(\hat{x})^{k+2} [\partial_t \Pi_t(y) + \mathbf{w}^h(t; y) \cdot [\nabla \Pi_t(y)] - \mathbf{w}^h(t; y)] & \text{if } x \notin \sigma \\ \mathbf{w}^h(t; x) & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathbf{w}^h(t; x) - \tilde{\mu}(\hat{x})^{k+2} [(\mathbf{w}^h(t; y) - \mathbf{w}(t; \Pi_t(y))) \cdot \nu_\Gamma(\Pi_t(y)) \nu_\Gamma(\Pi_t(y)) + d_\Gamma(t; y)T(y)] & \text{if } x \notin \sigma, \\ \mathbf{w}^h(t; x) & \text{otherwise,} \end{cases} \end{aligned}$$

with $T(x) := \partial_t[\nu_\Gamma(\Pi_t(x))] + \mathbf{w}^h(t; x) \cdot \nabla[\nu_\Gamma(\Pi_t(x))]$. We have used the tubular neighbourhood decomposition (2.4) and the following formulae:

$$\partial_t d_\Gamma(t; x) = -\mathbf{w}(t; \Pi_t(x)) \cdot \nu_\Gamma(\Pi_t(x)), \quad \nabla d_\Gamma(t; x) = \nu_\Gamma(\Pi_t(x)), \quad x \in \mathcal{N}_\Gamma,$$

see [34, Sec. 2]. This gives us the following lemma:

Lemma 3.8. *For h small enough, the map $\Lambda^h(t; \cdot)|_{K_i(t)}$ is a $C^{k+1}(K_i(t); \mathbb{R}^d)$ element-wise diffeomorphism with image $\Lambda^h(t; \Omega_i^h(t)) = \overline{\Omega}_i(t)$. Moreover, define the following:*

$$\mathcal{J}_i^l(t) := \{\Lambda^h(t; K_i(t)) \mid K_i(t) \in \mathcal{J}_i^h(t)\}, \quad \mathcal{J}^l(t) := \mathcal{J}_1^l(t) \cup \mathcal{J}_2^l(t).$$

Then $\mathcal{J}_1^l(t), \mathcal{J}_2^l(t), \mathcal{J}^l(t)$ define a uniform k -regular evolving subdivision of $\overline{\Omega}_1(t), \overline{\Omega}_2(t), \overline{\Omega}$, respectively.

This follows from [22, Lem. 8.12] and the fact that facets are mapped to their evolving equivalent can be shown in the exact same way as in Lem. 3.2. A chart representing the full set-up is given in Fig. 3.4.

For a function $v^h \in H^h(t)$, the lift is denoted by $(\cdot)^l : H^h(t) \rightarrow H(t)$ and defined as follows:

$$v^{h,l}(x) := (v_1^h(t; [\Lambda^h(t; x)]^{-1}), v_2^h(t; [\Lambda^h(t; x)]^{-1})).$$

Its inverse will be labelled by $(\cdot)^{-l}$, i.e. $(v^{h,l})^{-l} = v^h$. Since $\Lambda^h(t; \cdot)|_{\overline{\Omega}_i^h(t)} \in W_T^{k+1, \infty}(\mathcal{J}_i^h(t); \mathbb{R}^d)$, with norm uniformly bounded in h , and invertible, via a similar change of variable method as Lem. 2.12 we have:

$$\begin{aligned} c_1 \|v^{h,l}\|_{H(t)} &\leq \|v^h\|_{H^h(t)} \leq c_2 \|v^{h,l}\|_{H(t)} & \text{for } v^h \in H^h(t) \\ c_1 \|v^{h,l}\|_{V(t)} &\leq \|v^h\|_{V^h(t)} \leq c_2 \|v^{h,l}\|_{V(t)} & \text{for } v^h \in V^h(t). \end{aligned}$$

We define the analogous flow $\Phi_t^l : \overline{\Omega}_i(0) \rightarrow \overline{\Omega}_i(t)$ defined via the equation: $\Phi_t^l(t; \Lambda^h(0; x)) = \Lambda^h(t; \Phi_i^h(t; x))$. By the invertibility of Λ^l , this defines a flow, for which we can associate a push-forward map ϕ_t^l and inverse ϕ_{-t}^l as before. Note that this flow satisfies all properties B1 to B3 and D1 to D3 on the triplet

$V(t) \subset H(t) \subset V^*(t)$ and therefore can be equipped with its own material derivative ∂_t^l :

$$\partial_t^l \zeta := \phi_t^l \frac{\partial}{\partial t} \phi_{-t}^l \zeta,$$

for $\zeta \in C_{(H, \phi_t^l)}^1$ (we make the flow ϕ_t^h explicit in the label for the space $C_{(H, \phi_t^l)}^1$, so as to distinguish the space from C_H^1). Moreover, it is shown in [22, Lem. 3.5]:

$$\partial_t^l \eta^{h,l} = (\partial_t^h \eta^h)^l, \quad (3.8)$$

for $\eta^h \in C_{H^h}^1$.

3.4. Finite Element Spaces. Let $\alpha(t)$ be a Lagrangian node and $\mathcal{J}_i(\alpha(t))$ be the set of elements in $K_i(t) \in \mathcal{J}_i(t)$ such that $\alpha(t) \in K_i(t)$, and let $\mathcal{N}_i^h(t)$ be the global set of all Lagrangian nodes in $\mathcal{J}_i^h(t)$. We introduce the finite dimensional subspace:

$$\begin{aligned} \mathcal{S}_i^h(t) := & \left\{ \chi_i^h = (\chi_i^h)_{K_i(t) \in \mathcal{J}_i^h(t)} \in \prod_{K_i(t) \in \mathcal{J}_i^h(t)} \{ \widehat{\chi} \circ F_{K_i(t)}^{-1} : \widehat{\chi} \in \widehat{P}_k \} : \right. \\ & \left. \chi_i^h|_K(\alpha(t)) = \chi_i^h|_{K'}(\alpha(t)) \text{ for all } K_i(t), K_i'(t) \in \mathcal{J}_i(\alpha(t)), \forall \alpha(t) \in \mathcal{N}_i^h(t) \right\}. \end{aligned}$$

Combining two copies of the space yields the adequate solution space:

$$\begin{aligned} \mathcal{S}^h(t) := & \{ \eta^h = (\eta_1^h, \eta_2^h) \in \mathcal{S}_1^h(t) \times \mathcal{S}_2^h(t) \mid \chi_1^h(\alpha(t)) = \chi_2^h(\alpha(t)) \\ & \text{for all } \alpha(t) \in \Gamma^h(t) \cap \mathcal{N}_1^h(t) \text{ and } \chi_2^h(\alpha(t)) = 0 \quad \forall \alpha(t) \in \partial\Omega \cap \mathcal{N}_2^h(t) \}, \end{aligned}$$

and we equip $\mathcal{S}^h(t)$ with the same norm as $V^h(t)$.

Lemma 3.9. $(\mathcal{S}^h(t), \phi_t^h)|_{t \in I}$ form a compatible pair.

Proof. Since both the Lagrangian nodes and polynomials are evolved via Φ_t^h , one has by the definition of $\mathcal{S}(t)$, $\phi_t(\mathcal{S}(0)) = \mathcal{S}(t)$. Showing the remaining criterion for compatibility can be done in the same way as in Lem. 2.12. \square

Hence the moving spaces $L_{\mathcal{S}^h}^2$ is well defined.

The lifted solution space can now be defined as:

$$\mathcal{S}^l(t) := \{ \chi^{h,l} \mid \chi^h \in \mathcal{S}^h(t) \}.$$

The interpolation operator onto $\mathcal{S}^l(t)$, $I^l : C(\Omega) \rightarrow \mathcal{S}^l(t)$ can also be defined in a similar way:

$$I^l(\eta)|_{K(t)} := \sum_{\{j: \chi^j \in P^k(t)\}} \eta(\alpha_{K(t)}^{j,l}) \chi^{j,l}.$$

where $\{\alpha_{K(t)}^{j,l}\}_{j=1}^{N(k)}$ are the lifted Lagrangian Nodes.

The following variant of the approximation lemma holds:

Lemma 3.10 ([22], Lem. 8.21). *We have the estimates:*

$$\begin{aligned} \|w - I^l w\|_{H(t)} + h \|w - I^l w\|_{V(t)} &\leq ch^{k+1} \|w\|_{Z_k(t)}, & \text{for } w \in Z_k(t), \\ \|w - I^l w\|_{H(t)} + h \|w - I^l w\|_{V(t)} &\leq ch^2 \|w\|_{Z_1(t)}, & \text{for } w \in Z_1(t). \end{aligned}$$

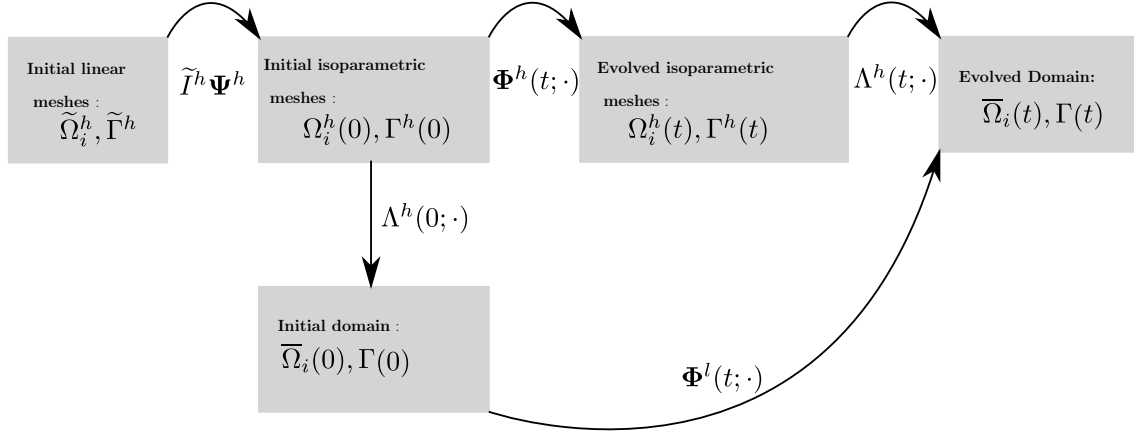


FIGURE 3.4. Schematic of the setup used. $\Lambda^h(t; x)$ might be needed, depending on the problem, to define the discrete data. However once the discrete problem is known, only the knowledge of $\Omega_i^h(0)$, $\Gamma^h(0)$ and $\Phi^h(t; \cdot)$ are needed to calculate the discrete solution $U^h(t; \cdot)$. Φ^l is only needed in the analysis of theoretical error estimates.

4. EVOLVING FINITE ELEMENT METHOD

4.1. **Scheme.** For any $U^h, \zeta^h \in V^h(t)$, let:

$$\begin{aligned} m^h(t; U^h, \zeta^h) &:= \sum_{i=1}^2 \int_{\Omega_i^h(t)} U_i^h \zeta_i^h, \\ a^h(t; U^h, \zeta^h) &:= \sum_{i=1}^2 \int_{\Omega_i^h(t)} \mathcal{A}_i(t; \Lambda_i^h(t; x)) \nabla U_i^h \cdot \nabla \zeta_i^h + [\mathcal{B}(t; \Lambda_i^h(t; x)) - \mathbf{w}^h] \cdot \nabla U^h \zeta^h \\ &\quad + [\mathcal{C}(t; \Lambda_i^h(t; x)) - \nabla \cdot \mathbf{w}^h(t; x)] U^h \zeta^h, \\ l^h(t; \zeta^h) &:= (f^{-l} J^h, \zeta^h)_{H^h(t)} + (g^{-l} \mu^h, \zeta^h)_{L^2(\Gamma^h(t))}, \end{aligned}$$

where J^h, μ^h are the discrete Jacobians with respect to the lift maps $\Lambda^h(t; x)|_{\Gamma^h(t)}$, and by regularity of Λ^h , are of class $C^k(K_i(t); \mathbb{R}^d)$, $C^k(E(t); \mathbb{R}^d)$, $\forall K_i(t) \in \mathcal{J}_i^h(t)$, $\forall E(t) \in \mathcal{J}_\Gamma^h(t)$, respectively.

The finite element method is to find $U^h(t) \in \mathcal{S}^h(t)$ satisfying the discrete variational problem:

$$m^h(t; \partial_t^h U^h, \zeta^h) + a^h(t; U^h, \zeta^h) + \lambda^h(t; U^h, \zeta^h) = l^h(t; \zeta^h) \quad \forall \zeta^h \in L_{\mathcal{S}^h}^2, \forall t \in I, \quad (4.1)$$

$$U^h(0) = U_0^h := \sum_{j=1}^{\dim(\mathcal{S}^h)} (u_0, \chi_j^h)_{H(0)} \chi_j^h.$$

Remark 4.1. It might not be practical to calculate $l^h(t; \cdot)$ for an arbitrary pair $(f, g) \in L_H^2 \times L_{\mathcal{V}_\Gamma}^2$ as it would have to be calculated via numerical integration. We will assume for the rest of the paper that it is possible to calculate these integrals exactly. See [14] for numerical integration on curved domains.

This formulation can be rearranged to a more useful form via the transport theorem with respect to the form $m^h(t; \cdot, \cdot)$:

$$\frac{d}{dt} m^h(t; U^h, \zeta^h) - m^h(t; U^h, \partial_t^h \zeta^h) + a^h(t; U^h, \zeta^h) = l^h(t; \zeta^h), \quad (4.2)$$

for $\zeta^h \in C_{\mathcal{S}^h}^1$. Moreover, by construction of the $l^h(t; \cdot)$ term, for a function $\eta^h \in H^h(t)$:

$$l(t; \eta^{h,l}) = l^h(t; \eta^h).$$

4.2. Well posedness of the finite element scheme.

Theorem 4.2. *There exists a unique solution to (4.1) with continuous bound:*

$$\sup_{t \in I} \|U^h\|_{H^h(t)}^2 + \int_0^T \|U^h\|_{V^h(t)}^2 \leq C(T) \left(\|U_0^h\|_{H^h(0)}^2 + \|f\|_{L_H^2} + \|g\|_{L_{V^*}^2} \right).$$

Proof. Substituting the Ansatz:

$$U^h(t; x) = \sum_{j=1}^{\dim(\mathcal{S}_h)} \alpha_j(t) \chi_j^h(t; x),$$

where $\{\chi_j^h(t; x)\}_{j=1}^{\dim(\mathcal{S}_h)}$ are the basis functions of the evolving solution space $L_{\mathcal{S}^h}^2$. We refer to [22, Lem. 3.1] for proof of the *transport property*:

$$\partial_t^h \chi_j^h = 0 \quad \forall j \in 1, \dots, N(k).$$

Then the problem can be restated as the finite dimensional problem:

$$\begin{aligned} \frac{d}{dt}(\mathbf{M}(t)\alpha(t)) + \mathbf{A}(t)\alpha(t) &= \mathbf{L}(t) \\ \alpha(0) &= \alpha_0 \end{aligned}$$

where:

$$\begin{aligned} \alpha(t) &= (\alpha_1(t), \dots, \alpha_{N(k)}(t)), & [\mathbf{M}(t)]_{j,k} &= m^h(t; \chi_j^h, \chi_k^h), \\ [\mathbf{A}(t)]_{j,k} &= a^h(t; \chi_j^h, \chi_k^h), & \mathbf{L}(t) &= (l^h(t; \chi_1^h), \dots, l^h(t; \chi_{N(k)}^h)). \end{aligned}$$

Note that $\mathbf{M}(t)$ is a Gram matrix (and hence invertible). Hence, by use of standard ODE theory (see [46, Sec. 1.6]), there exists a solution $\alpha(t) \in \mathcal{W}^{1,1}(\mathbb{R}; \mathbb{R}^{N(k)})$. The uniform bound and uniqueness follows from testing with U^h and using the transport theorem. \square

5. ERROR BOUND

The main result of this article is the following optimal order error bound.

Theorem 5.1. *If the solution to (4.1) is of regularity $u \in W(Z_k, Z_k) \cap L_{Z_k}^\infty$ with uniform bound:*

$$\|u\|_{L_{Z_k}^\infty} + \|u\|_{W(Z_k, Z_k)} \leq C_u,$$

then there exists a constant \mathcal{C} depending on C_u such that the following holds:

$$\sup_{t \in I} \|u - U^{h,l}\|_{H(t)}^2 + h^2 \int_0^T \|u - U^{h,l}\|_{V(t)}^2 \leq c \|u_0 - u_0^{h,l}\|_{H(0)}^2 + ch^{2k+2}(\mathcal{C}).$$

Note that under the assumption of there existing a moving space equivalence on $W(Z_k, Z_k)$, $u \in L_{Z_k}^\infty$ automatically (see Lem. 2.7) and hence it only suffices to assume $u \in W(Z_k, Z_k)$. In the next section we set out preliminary approximation results and then prove the error bound in the subsequent section.

In the next two subsections we introduce necessary tools from [22, Sec. 3.3] in order to obtain suitable orders of convergence. We will assume that for each space $W(Z_k, Z_k)$, there exists a moving space equivalence with $\mathcal{W}(Z_k(0), Z_k(0))$, see Def. 2.5. This only requires the flow map Φ_t to be regular enough and in particular in guaranteed if Φ_t is smooth.

5.1. **Geometric perturbations.** Let:

$$\begin{aligned} b^l(t; w, v) &:= \frac{d}{dt}[a(t; w, v)] - a(t; \partial_t^l w, v) - a(t; w, \partial_t^l v), & w, v \in W(V, V), \\ b^h(t; w^h, v^h) &:= \frac{d}{dt}[a^h(t; w^h, v^h)] - a(t; \partial_t^h w^h, v^h) - a(t; w^h, \partial_t^h v^h), & w^h, v^h \in W(V^h, V^h), \end{aligned}$$

a.e for $t \in I$. Similarly as (2.11), these bilinear forms can be calculated explicitly and satisfy:

$$|b^l(t; w, v)| \leq c \|w\|_{V(t)} \|v\|_{V(t)}, \quad |b^h(t; w^h, v^h)| \leq c \|w^h\|_{V(t)} \|v^h\|_{V(t)}, \quad \forall w, v \in V(t), \forall w^h, v^h \in V^h(t),$$

for some constant c independent of t and h . Define $\lambda^l(t; \cdot, \cdot)$ to be the bilinear form of Def. 2.3 with respect to the flow Φ^l , which can be calculated to be:

$$\lambda^l(t; v, w) = \sum_{i=1}^2 \int_{\Omega_i(t)} \nabla \cdot \tilde{\mathbf{w}} v w, \quad (5.1)$$

where $\tilde{\mathbf{w}}$ is defined as:

$$\tilde{\mathbf{w}}(t; x) = \frac{\partial}{\partial t} \Phi^l(t; y)|_{y=\Phi^l(-t; x)}. \quad (5.2)$$

See [22, Lem. 8.15].

Lemma 5.2 ([22], Lem. 8.16). *The lift satisfies the following:*

$$\begin{aligned} \sup_{t \in I} \|\nabla \Lambda^h(t; \cdot) - \mathbf{I}\|_{L^\infty(\Omega_i^h(t))} &\leq ch^k, \\ \sup_{t \in I} \|\partial_t^h \nabla \Lambda^h(t; \cdot)\|_{L^\infty(\Omega_i^h(t))} &\leq ch^k, \end{aligned}$$

and the Jacobian $J^h := \sqrt{\det[\nabla \Lambda^h]^T \nabla \Lambda^h}$ satisfies:

$$\sup_{t \in I} \|J^h(t; \cdot) - 1\|_{L^\infty(\Omega_i^h(t))} \leq ch^k.$$

Then the following holds for the bilinear forms introduced in (2.9) and (2.11):

Proposition 5.3. *There exists a constant $c > 0$ such that for almost all $t \in I$ and for all $w^h, v^h \in V^h(t)$, $w^{h,l}, v^{h,l} \in V(t)$ the following error bounds hold:*

$$|m(t; w^{h,l}, v^{h,l}) - m^h(t; w^h, v^h)| \leq ch^{k+1} \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}, \quad (\text{P1})$$

$$|\lambda(t; w^{h,l}, v^{h,l}) - \lambda^h(t; w^h, v^h)| \leq ch^{k+1} \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}, \quad (\text{P2})$$

$$|\lambda^l(t; w^{h,l}, v^{h,l}) - \lambda(t; w^{h,l}, v^{h,l})| \leq ch^k \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}, \quad (\text{P3})$$

$$|a(t; w^{h,l}, v^{h,l}) - a^h(t; w^h, v^h)| \leq ch^k \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}, \quad (\text{P4})$$

$$|b^l(t; w^{h,l}, v^{h,l}) - b^h(t; w^h, v^h)| \leq ch^k \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}, \quad (\text{P5})$$

$$|b^l(t; w^{h,l}, v^{h,l}) - b(t; w^{h,l}, v^{h,l})| \leq ch^k \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}. \quad (\text{P6})$$

For $\eta, \zeta \in Z_1(t)$ with inverse lifts η^{-l}, ζ^{-l} :

$$|a(t; \eta, \zeta) - a^h(t; \eta^{-l}, \zeta^{-l})| \leq ch^{k+1} \|\eta\|_{Z_1(t)} \|\zeta\|_{Z_1(t)}, \quad (\text{P4}')$$

$$|b^l(t; \eta, \zeta) - b^h(t; \eta^{-l}, \zeta^{-l})| \leq ch^{k+1} \|\eta\|_{Z_k(t)} \|\zeta\|_{Z_k(t)}. \quad (\text{P5}')$$

For $\eta \in C_{Z_k}^1$ and $\zeta \in Z_1(t)$, with inverse lifts η^{-l} and ζ^{-l} :

$$|a(t; \partial_t^l \eta, \zeta) - a^h(t; \partial_t^h \eta^{-l}, \zeta^{-l})| \leq ch^{k+1} (\|\eta\|_{Z_1(t)} + \|\partial_t^\bullet \eta\|_{Z_1(t)}) \|\zeta\|_{Z_1(t)}. \quad (\text{P7})$$

The material derivatives satisfy

$$\|\partial_t^l \zeta - \partial_t^\bullet \zeta\|_{H(t)} \leq ch^{k+1} \|\zeta\|_{V(t)} \text{ for } \zeta \in C_V^1, \quad (\text{P8})$$

$$\|\partial_t^l \zeta - \partial_t^\bullet \zeta\|_{V(t)} \leq ch^k \|\zeta\|_{Z_1(t)} \text{ for } \zeta \in C_{Z_1}^1. \quad (\text{P9})$$

Proof. In [22, Lem. 8.23 and 8.24], these estimates are proven on a single evolving domain $\Omega(t)$ with $V(t) = H^1(\Omega(t))$. However, almost the same arguments cover our case. We will only show this for (P1), (P4) and (P9), but the same method can be applied for the remaining claims.

(P1): Let $J^h := \sqrt{\det[\nabla \Lambda^h]^T \nabla \Lambda^h}$ be Jacobian resulting from switching from $\Omega_i(t)$ to $\Omega_i^h(t)$. Then, the lift itself differs from the identity only when x is in an interface element: let

$$M := \{x \in \Omega_i^h(t) : J^h(x) \neq 1\} \subset \{x \in \Omega_i(t) : |d_\Gamma(t; x)| \leq h\},$$

and let $M^l := \{\Lambda^h(t; x) | x \in M\} \subset \{x \in \Omega_i(t) : |d_\Gamma(t; x)| \leq h\}$. For $w_i^h \in H^1(\Omega_i(t))$:

$$\left| \int_{\Omega_i(t)} w_i^{h,l} \cdot v_i^{h,l} - \int_{\Omega_i^h(t)} w_i^h \cdot v_i^h \right| = \left| \int_M w_i^h \cdot v_i^h [J^h - 1] \right| \leq ch^k \|w_i^{h,l}\|_{L^2(M^l)} \|v_i^{h,l}\|_{L^2(M^l)}.$$

Then, via the Narrow-Band trace inequality, see [21, Lem. 4.10], we see:

$$\|w_i^{h,l}\|_{L^2(M^l)} \leq ch^{1/2} \|w_i^{h,l}\|_{H^1(\Omega_i(t))},$$

and hence:

$$\begin{aligned} |m(t; w^{h,l}, v^{h,l}) - m^h(t; w^h, v^h)| &= \left| \sum_{i=1}^2 \int_{\Omega_i(t)} w_i^{h,l} \cdot v_i^{h,l} - \int_{\Omega_i^h(t)} w_i^h \cdot v_i^h \right|, \\ &\leq ch^{k+1} \sum_{i=1}^2 \|w_i^{h,l}\|_{H^1(\Omega_i(t))} \|v_i^{h,l}\|_{H^1(\Omega_i(t))}, \\ &\leq ch^{k+1} \|w^{h,l}\|_{V(t)} \|v^{h,l}\|_{V(t)}. \end{aligned}$$

(P4): Domain-wise (using the shorthand $\Lambda^h(t; x) = \Lambda^h$):

$$\begin{aligned} &\int_{\Omega_i(t)} \mathcal{A}_i(t; x) \nabla w_i^{h,l} \cdot \nabla v_i^{h,l} + [\mathcal{B}_i(t; x) - \mathbf{w}] \cdot \nabla w_i^{h,l} v_i^{h,l} + [\mathcal{C}_i(t; x) - \nabla \cdot \mathbf{w}] w_i^{h,l} v_i^{h,l} \\ &= \int_{\Omega_i^h(t)} [\mathcal{A}_i(t; \Lambda^h) \nabla \Lambda^h \nabla w_i^h \cdot \nabla \Lambda^h \nabla v_i^h + [\mathcal{B}_i(t; \Lambda^h) - \mathbf{w}(t; \Lambda^h)] \cdot \nabla \Lambda^h \nabla w_i^h v_i^h \\ &\quad + [\mathcal{C}_i(t; \Lambda^h) - \text{Tr}(\nabla \Lambda^h \nabla \mathbf{w})] w_i^h v_i^h] J^h. \end{aligned}$$

By use of both Lem. 3.10 and 5.2, P4 follows in the same way as (P1).

(P9): Explicitly expanding both material derivatives:

$$\partial_t^\bullet \zeta = \partial_t \zeta + \mathbf{w} \cdot \nabla \zeta = \partial_t^l \zeta + [\mathbf{w} - \tilde{\mathbf{w}}] \cdot \nabla \zeta. \quad (5.3)$$

Using the definition of Φ_t^l and (5.2), element-wise, for $x \in K(t)$:

$$\partial_t^h \Lambda^h(t; x) = \phi_t^h \partial_t \Lambda^h(t; \Phi^h(t; x)) = \phi_t^h \partial_t \Phi^l(t; \Lambda^h(0; x)) = \phi_t^h \tilde{\mathbf{w}}(t; \Phi^l(t; \Lambda^h(0; x))) = \tilde{\mathbf{w}}(t; \Lambda^h(t; x)).$$

Rewriting (5.3):

$$\begin{aligned} \partial_t^\bullet \zeta - \partial_t^l \zeta &= [\mathbf{w} - \partial_t^h \Lambda^h(t; z)]|_{z=[\Lambda^h(t; x)]^{-1}} \cdot \nabla \zeta, \\ &= [\mathbf{w} - I^l \mathbf{w} + I^l \mathbf{w} - \partial_t^h \Lambda^h(t; z)]|_{z=[\Lambda^h(t; x)]^{-1}} \cdot \nabla \zeta, \end{aligned} \quad (5.4)$$

via [Lem. 3.10](#):

$$\|\mathbf{w} - I^l \mathbf{w}\|_{V(t)} \leq ch^k \|\mathbf{w}\|_{Z^k(t)}, \quad (5.5)$$

as for the remaining term, $\partial_t^h \Lambda^h(t; y)$, using [\(3.7\)](#):

$$\begin{aligned} I^l \mathbf{w} - \partial_t^h \Lambda^h(t; z)|_{z=[\Lambda^h(t; x)]^{-1}} &= [\mathbf{w}^h(t; z) - \partial_t^h \Lambda^h(t; z)]_{z=[\Lambda^h(t; x)]^{-1}} \\ &= \begin{cases} -\tilde{\mu}(\hat{x})^{k+2} [(\mathbf{w}^h(t; y) - \mathbf{w}(t; \Pi_t(y))) \cdot \nu_\Gamma(\Pi_t(y)) \nu_\Gamma(\Pi_t(y)) - d_\Gamma(t; y) T(y)] & \text{if } x \notin \sigma, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Via the use of standard geometric estimates, see [\[22, Lem. 8.16 and 9.10\]](#) and the fact that \mathbf{w}^h is the interpolant of \mathbf{w} , we infer that:

$$\|\mathbf{w}^h(t; z) - \partial_t^h \Lambda^h(t; z)\|_{W^{1,\infty}(K(t))} \leq ch^k. \quad (5.6)$$

Combining [\(5.4\)](#) to [\(5.6\)](#) yields [\(P9\)](#). \square

5.2. Ritz Projection. We set

$$a_\kappa(t; w, v) := \kappa m(t; w, v) + a(t; w, v) \quad (5.7)$$

and observe that

$$\begin{aligned} a_\kappa(t; v, v) &\geq \left(\gamma - \frac{\epsilon \|\mathcal{B}_i - \mathbf{w}\|_{L^\infty(\Omega)}}{2} \right) \|\nabla v\|_{H(t)}^2 \\ &\quad + \left(\kappa - \|\nabla \cdot \mathbf{w}\|_{L^\infty(\Omega)} - \|\mathcal{C}_i\|_{L^\infty(\Omega)} - \frac{\|\mathcal{B}_i - \mathbf{w}\|_{L^\infty(\Omega)}}{2\epsilon} \right) \|v\|_{H(t)}^2, \end{aligned}$$

where γ is a lower bound for the eigenvalues of \mathcal{A}_i [\(2.10\)](#). Thus taking ϵ sufficiently small and κ sufficiently large, we may choose κ depending only on the data to ensure that the bounded bilinear form $a_\kappa(t; \cdot, \cdot)$ is strictly coercive.

Similarly, we define

$$a_\kappa^h(t; w_h, v_h) := \kappa m^h(t; w_h, v_h) + a^h(t; w_h, v_h),$$

which is also coercive provided κ is large enough, independently of h , by the same argument.

The *Ritz projection* $\Pi^h: V(t) \rightarrow \mathcal{S}^h(t)$ is defined as the solution to:

$$a_\kappa^h(t; \Pi^h(\eta), v^h) = a_\kappa(t; \eta, v^{h,l}) \quad \forall v^h \in \mathcal{S}^h(t), \quad (5.8)$$

and $\pi^h \eta := (\Pi^h \eta)^l$. By the coercivity and boundedness of a_κ^h , this gives us a uniformly bounded and linear operator $\Pi^h: V(t) \rightarrow \mathcal{S}^h(t)$. Moreover, it is further proven in [\[22, Lem. 3.9\]](#), that $\Pi^h \eta \in C_{\mathcal{S}^h}^1$ if $\eta \in C_V^1$ and it follows by use of the same method that $\Pi^h \eta \in C_{\mathcal{S}^h}^0$ if $\eta \in C_V^0$.

Lemma 5.4. *The Ritz projection can be extended as a continuous linear operator $\Pi^h(\cdot): L_V^2 \rightarrow L_{\mathcal{S}^h}^2$. Moreover, if $\eta \in W(V, V)$, then $\Pi^h(\eta) \in W(\mathcal{S}_h, \mathcal{S}_h) = \{v^h \in L_{\mathcal{S}^h}^2, \partial_t^h v^h \in L_{\mathcal{S}^h}^2\}$. In particular, $\Pi^h(\cdot): W(V, V) \rightarrow W(\mathcal{S}_h, \mathcal{S}_h)$ is linear and bounded uniformly in h .*

Proof. Integrating [\(5.8\)](#), we see, that for all $v^h \in L_{\mathcal{S}^h}^2$:

$$\int_0^T a_\kappa^h(t; \Pi^h(\eta), v^h) dt = \int_0^T a_\kappa(t; \eta, v^{h,l}) dt.$$

By point-wise coercivity in time of a_κ^h , we also get the coercivity over $L_{\mathcal{S}^h}^2$. Since $L_{\mathcal{S}^h}^2$ is a closed subspace of a Hilbert space, and $\int_0^T a_\kappa(t; \eta, (\cdot)^l) dt$ defines a bounded linear functional on $L_{\mathcal{S}^h}^2$, via the standard use of Lax-Milgram, there exists a unique solution, labelled $\Pi^h(\eta)$ and we achieve the bound:

$$\|\Pi^h(\eta)\|_{L_{\mathcal{S}^h}^2} \leq \sup_{v^h \in L_{\mathcal{S}^h}^2, \|v^h\|=1} \int_0^T a_\kappa(t; \eta, v^{h,l}) dt \leq c\|\eta\|_{L_V^2}. \quad (5.9)$$

Hence $\Pi^h(\cdot)$ is continuous. Note that the bound in (5.9) can be taken to be independent of h , this is due to the fact that both the bilinear form a_κ and the lift map $(\cdot)^l$ are both bounded independently of h .

To show the second claim, assume $\eta \in W(V, V)$ and set ζ as the solution to:

$$\int_0^T a_\kappa^h(t; \zeta, v^h) dt = \int_0^T a_\kappa(t; \partial_t^l \eta, v^{h,l}) - b_\kappa^h(t; \Pi^h \eta, v^h) + b_\kappa^l(t; \eta, v^{h,l}) dt, \quad \forall v^h \in L_{\mathcal{S}^h}^2. \quad (5.10)$$

The via the same argument as before, there exists a unique $\zeta \in L_{\mathcal{S}^h}^2$ solving (5.10) with bound:

$$\|\zeta\|_{L_{\mathcal{S}^h}^2} \leq c\|\eta\|_{W(V,V)}, \quad (5.11)$$

and similarly as before, the bound in equation (5.11) is independent of h . Since $\eta \in W(V, V)$, it is also in C_V^0 by Lem. 2.7, $\eta \in C_V^0$ and therefore $\Pi(\eta) \in C_{\mathcal{S}^h}^0$.

Define:

$$w^h := \phi_t^h \int_0^t \phi_{-s}^h \zeta(s) ds + \phi_t^h \Pi^h(\eta)(0).$$

Via the isomorphism lemma (Lem. 2.2), $\phi_{-s}^h \zeta(s) \in L^2(I; \mathcal{S}^h(0))$, the standard Bochner space and therefore is Bochner integrable. Since $\mathcal{S}^h(0)$ is a closed linear subspace, the definite Bochner integral of a function inside $\mathcal{S}^h(0)$ remains in $\mathcal{S}^h(0)$ for all $t \in I$. Using the isomorphism again, we see that $w^h \in L_{\mathcal{S}^h}^2$. We will show that $w^h = \Pi^h \eta$ which will show the second claim. Indeed, $w^h \in W(\mathcal{S}^h, \mathcal{S}^h)$ with $\partial_t^h w^h = \zeta$. Substituting this back into (5.10):

$$\int_0^T a_\kappa^h(t; \partial_t^h w, v^h) dt = \int_0^T a_\kappa(t; \partial_t^l \eta, v^{h,l}) - b_\kappa^h(t; \Pi^h \eta, v^h) + b_\kappa^l(t; \eta, v^{h,l}) dt, \quad \forall v^h \in L_{\mathcal{S}^h}^2, \quad (5.12)$$

using the definition of b_κ^h , we see that, for $v^h \in \mathcal{W}(\mathcal{S}^h, \mathcal{S}^h)$ with $v^h(0) = v^h(T) = 0$:

$$\begin{aligned} \int_0^T a_\kappa^h(t; \partial_t^h w^h, v^h) dt &= \int_0^T -b_\kappa^h(t; w^h, v^h) - a_\kappa^h(t; w^h, \partial_t^h v^h) dt, \\ \int_0^T a_\kappa(t; \partial_t^l \eta, v^{h,l}) dt &= \int_0^T -b_\kappa^l(t; \eta, v^{h,l}) - a_\kappa(t; \eta, [\partial_t^h v^h]^l) dt, \end{aligned}$$

using the commutation properties of the material derivatives ∂_t^h and ∂_t^l (see (3.8)). Substituting these expression back into (5.12):

$$\int_0^T a_\kappa(t; \eta, [\partial_t^h v^h]^l) - a_\kappa^h(t; w^h, \partial_t^h v^h) dt = \int_0^T b_\kappa^h(t; w^h - \Pi^h \eta, v^h) dt. \quad (5.13)$$

Using the definition or the Ritz projection (5.8), we arrive at:

$$\int_0^T a_\kappa^h(t; \Pi^h \eta - w^h, \partial_t^h v^h) dt = \int_0^T b_\kappa^h(t; w^h - \Pi^h \eta, v^h) dt. \quad (5.14)$$

Testing this equation with $v^h(t) = \varphi(t)\psi^h(t)$ with $\varphi(t) \in \mathcal{D}(I)$ and $\psi^h(t) \in W(\mathcal{S}^h, \mathcal{S}^h)$, we see:

$$\int_0^T \varphi(t) a_\kappa^h(t; \Pi^h \eta - w^h, \partial_t^h \psi^h) + \varphi'(t) a_\kappa^h(t; \Pi^h \eta - w^h, \psi^h) dt = \int_0^T \varphi(t) b_\kappa^h(t; w^h - \Pi^h \eta, \psi^h) dt.$$

Then, via the fundamental lemma of variational calculus, see [33, Thm. 1.2.1 and Lem. 1.2.1], and the fact that $a_\kappa^h(t; \Pi^h \eta - w^h, \psi^h)$ is continuous as $\Pi^h(\eta) - w^h, \psi^h \in C_{\mathcal{S}^h}^0$, we get that for all $t \in I$:

$$a_\kappa^h(t; \Pi^h \eta - w^h, \psi^h) = \int_0^t b_\kappa^h(s; \Pi^h \eta - w^h, \psi^h) + a_\kappa^h(s; \Pi^h \eta - w^h, \partial_s^h \psi^h) ds \quad (5.15)$$

Fix $t \in I$ and test with $\psi^h(s) = \phi_s^h \phi_{-t}^h(\Pi^h \eta - w^h)$, we see that $\partial_s^h \psi^h(s) = 0$ and hence, via the coercivity of a_κ and compatibility:

$$\begin{aligned} \|\Pi^h \eta - w^h\|_{\mathcal{S}^h(t)}^2 &\leq c \int_0^t b_\kappa^h(s; \Pi^h \eta - w^h, \psi^h) ds, \\ &\leq c \int_0^t \|\Pi^h \eta - w^h\|_{\mathcal{S}^h(s)} \|\Pi^h \eta - w^h\|_{\mathcal{S}^h(t)} ds. \end{aligned}$$

By use of Young's inequality:

$$\|\Pi^h \eta - w^h\|_{\mathcal{S}^h(t)}^2 \leq c \int_0^t \|\Pi^h \eta - w^h\|_{\mathcal{S}^h(s)}^2 ds.$$

This holds for arbitrary point $t \in I$ and hence can be repeated to see that this holds for all of I . By use of Grönwall's inequality it must be that:

$$\|\Pi^h \eta - w^h\|_{\mathcal{S}^h(t)}^2 = 0,$$

and hence $\Pi^h \eta = w^h$ and therefore $\Pi^h \eta \in W(\mathcal{S}^h, \mathcal{S}^h)$. As for the bound on $\Pi^h \eta$, we see that, since $\partial_t^h w^h = \partial_t^h \Pi^h(\eta) = \zeta$, using (5.9) and (5.11) we see that $\|\Pi^h(\eta)\|_{W(\mathcal{S}^h, \mathcal{S}^h)} \leq c\|\eta\|_{W(V, V)}$ independently of h . \square

Note that $a_\kappa(t; \cdot, \cdot)$ and $a_\kappa^h(t; \cdot, \cdot)$ satisfy all the same estimates as $a(t; \cdot, \cdot)$ and $a^h(t; \cdot, \cdot)$ in (P1)–(P9).

Remark 5.5. We note that via the commutative properties of the material derivatives (see (3.8)) $[\partial_t^h \Pi^h(v)]^l = \partial_t^l \pi^h(v)$ for $v \in W(V, V)$. Hence we also conclude that the lifted Ritz map $\pi^h : W(V, V) \rightarrow W(\mathcal{S}_h^l, \mathcal{S}_h^l)$ is continuous as well, again uniformly in h .

Define the dual solution operators $\mathcal{R}_H : H(t) \rightarrow V(t)$ and $\mathcal{R}_{\mathcal{H}_\Gamma} : \mathcal{H}_\Gamma(t) \rightarrow V(t)$ to be the solutions to:

$$a_\kappa(t; w, \mathcal{R}_H(v)) = m(t; w, v) \quad \forall w \in V(t) \quad (5.16)$$

$$a_\kappa(t; w, \mathcal{R}_{\mathcal{H}_\Gamma}(v)) = (w, v)_{\mathcal{H}_\Gamma(t)} \quad \forall w \in V(t). \quad (5.17)$$

$\mathcal{H}_\Gamma(t), \mathcal{V}_\Gamma(t)$ are defined in Sec. 2.2. We aim to show that these operators satisfies the following regularity condition:

Lemma 5.6. *Assuming additional regularity on the data: $\mathcal{A}_i \in W^{1,\infty}(\Omega_i(t); \mathbb{R}^{d \times d})$, $\mathcal{B}_i \in W^{1,\infty}(\Omega_i(t); \mathbb{R}^d)$, $\mathcal{C}_i \in W^{1,\infty}(\Omega_i(t); \mathbb{R})$, the operators $\mathcal{R}_H, \mathcal{R}_{\mathcal{H}_\Gamma}$ satisfy the regularity bounds:*

$$\|\mathcal{R}_H(v)\|_{Z_1(t)} \leq c\|v\|_{H(t)} \quad (5.18)$$

$$\|\mathcal{R}_{\mathcal{H}_\Gamma}(v)\|_{Z_1(t)} \leq c\|v\|_{\mathcal{H}_\Gamma(t)}. \quad (5.19)$$

Proof. Writing (5.16) explicitly, we seek a solution $\mathcal{R}_H(v)$ to:

$$\sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{A}_i \nabla w_i \cdot \nabla \mathcal{R}_H(v) + [\mathcal{B}_i - \mathbf{w}] \cdot \nabla w_i \mathcal{R}_H(v) + [\mathcal{C}_i + \kappa - \nabla \cdot \mathbf{w}] w_i \mathcal{R}_H(v) = m(t; w, v). \quad (5.20)$$

We note, by increasing κ more if needs be, (5.20) is still coercive. Hence there exists a solution $\mathcal{R}_H(v) \in V(t)$ via use of the Babuska-Lax-Milgram theorem. Moreover, there exists a constant c independent of time such that:

$$\|\mathcal{R}_H(v)\|_{V(t)} \leq c\|v\|_{H(t)}.$$

To show the additional regularity, rearranging (5.20), we have:

$$a_\kappa(t; w, \mathcal{R}_H(v)) = a_\kappa(t; \mathcal{R}_H(v), w) + \sum_{i=1}^2 \int_{\Omega_i(t)} [\mathcal{B}_i - \mathbf{w}] \cdot ([\nabla w_i] \mathcal{R}_H(v) - [\nabla \mathcal{R}_H(v)] w_i). \quad (5.21)$$

The remaining term of (5.20) can be further rearranged as, using integration by parts and the continuity of $V(t)$ and \mathbf{w} across the interface:

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i(t)} [\mathcal{B}_i - \mathbf{w}] \cdot ([\nabla w_i] \mathcal{R}_H(v) - [\nabla \mathcal{R}_H(v)] w_i) \\ &= \int_{\Gamma(t)} \llbracket \mathcal{B} \rrbracket \cdot \nu w \mathcal{R}_H(v) - \sum_{i=1}^2 \int_{\Omega_i(t)} 2[\mathcal{B}_i - \mathbf{w}] \cdot [\nabla \mathcal{R}_H(v)] w_i + \nabla \cdot [\mathcal{B}_i - \mathbf{w}] \mathcal{R}_H(v) w_i. \end{aligned}$$

Hence, the solution to (5.16) also solves:

$$\begin{aligned} a_\kappa(t; \mathcal{R}_H(v), w) &= m(t; w, v) - \int_{\Gamma(t)} \llbracket \mathcal{B} \rrbracket \cdot \nu w \mathcal{R}_H(v) \\ &\quad + \sum_{i=1}^2 \int_{\Omega_i(t)} 2[\mathcal{B}_i - \mathbf{w}] \cdot [\nabla \mathcal{R}_H(v)] w_i + \nabla \cdot [\mathcal{B}_i - \mathbf{w}] \mathcal{R}_H(v) w_i, \quad \forall w \in V(t). \end{aligned} \quad (5.22)$$

Set $\mathcal{L}(v)$ to be the solution to:

$$\begin{aligned} a_\kappa(t; \mathcal{L}(v), w) &= - \underbrace{\int_{\Gamma(t)} \llbracket \mathcal{B} \rrbracket \cdot \nu w \mathcal{R}_H(v)}_{:= (\tilde{g}, w)_{\mathcal{H}_\Gamma(t)}} \\ &\quad + \underbrace{\sum_{i=1}^2 \int_{\Omega_i(t)} w_i v_i + 2[\mathcal{B}_i - \mathbf{w}] \cdot [\nabla \mathcal{R}_H(v)] w_i + \nabla \cdot [\mathcal{B}_i - \mathbf{w}] \mathcal{R}_H(v) w_i}_{:= (\tilde{f}, w)_{H(t)}}, \quad \forall w \in V(t). \end{aligned} \quad (5.23)$$

This solution exists by Thm. 2.16. We seek to show that first $\mathcal{L}(v) \in Z_1(t)$ and then that $\mathcal{R}_H(v) = \mathcal{L}(v)$. By use of Theorem 1 in [37], since $\tilde{f} \in H(t)$ and $\tilde{g} \in H^{1/2}(\Gamma(t))$, the solution to (5.23) is indeed in $Z_1(t)$ since the data $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)$ is regular enough and moreover:

$$\|\mathcal{L}(v)\|_{Z_1(t)} \leq c(\|v\|_{V_\Gamma(t)} + \|\mathcal{R}_H(v)\|_{V(t)}) \leq c\|v\|_{H(t)}$$

To show $\mathcal{R}_H(v) = \mathcal{L}(v)$, subtracting (5.23) from (5.22):

$$a_\kappa(t; \mathcal{R}_H(v) - \mathcal{L}(v), w) = 0.$$

Testing with $w = \mathcal{R}_H(v) - \mathcal{L}(v)$ and using coercivity yields $\mathcal{R}_H(v) = \mathcal{L}(v)$ and hence, we have:

$$\|\mathcal{R}_H(v)\|_{Z_1(t)} \leq c\|v\|_{H(t)}, \quad (5.24)$$

where the regularity constant c can be taken to be bounded on $[0, T]$ via the regularity of the flow.

The same argument follows for $\mathcal{R}_{\mathcal{H}_\Gamma}$ with

$$\begin{aligned} (\tilde{f}, w)_{H(t)} &= \sum_{i=1}^2 \int_{\Omega_i(t)} 2[\mathcal{B}_i - \mathbf{w}] \cdot [\nabla \mathcal{R}_{\mathcal{H}_\Gamma}(v)] w_i + \nabla \cdot [\mathcal{B}_i - \mathbf{w}] \mathcal{R}_H(v) w_i \\ (\tilde{g}, w)_{\mathcal{H}_\Gamma(t)} &= \int_{\Gamma(t)} wv - \llbracket \mathcal{B} \rrbracket \cdot \nu w \mathcal{R}_{\mathcal{H}_\Gamma}(v). \end{aligned} \quad \square$$

Lemma 5.7. *On top of the assumptions made in [Lem. 5.6](#), assume $\mathcal{A}_i \in C^2(\overline{\mathcal{Q}}_i; \mathbb{R}^{d \times d})$. For $w \in Z_k(t)$, $\eta := w - \pi^h w$ and $v \in Z_1(t)$ it holds that*

$$|b(t; \eta, v)| \leq c(\|\eta\|_{H(t)} + h\|\eta\|_{V(t)} + h^{k+1}\|w\|_{Z_k(t)})\|v\|_{Z_1(t)}. \quad (5.25)$$

Proof. We begin with the following estimate of the bilinear form $b(t; \cdot, \cdot)$, [\(2.11\)](#), for $\eta := w - \pi^h w$:

$$|b(t; \eta, v)| \leq c\|\eta\|_{H(t)}\|v\|_{H(t)} + \left| \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{D}_i^B(\mathbf{w}, \mathcal{B}_i, \eta_i, v_i) + \mathcal{D}_i^A(\mathbf{w}, \mathcal{A}_i, \eta_i, v_i) \right|.$$

For \mathcal{D}_i^A , integrating by parts yields:

$$\begin{aligned} & \left| \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{D}_i^A(\mathbf{w}, \mathcal{A}_i, \eta_i, v_i) \right| \\ & \leq \left| \int_{\Gamma(t)} \llbracket (\partial_t^\bullet \mathcal{A}_i + \nabla \cdot w \mathcal{A}_i - 2D(w_i, \mathcal{A}_i)) \nabla v \cdot \nu_\Gamma \rrbracket \eta \right| + \left| \sum_{i=1}^2 \eta_i \nabla \cdot ((\partial_t^\bullet \mathcal{A}_i + \nabla \cdot w \mathcal{A}_i - 2D(w_i, \mathcal{A}_i)) \nabla v_i) \right| \\ & \leq c \left(|\mathcal{A}_i|_{C^2(\overline{\mathcal{Q}}_i; \mathbb{R})}, |\nabla \mathbf{w}|_{C^1(\overline{\Omega}; \mathbb{R}^d)} \right) (\|\eta\|_{H(t)} + \|\eta\|_{\mathcal{V}_\Gamma^*(t)}) \|v\|_{Z_1(t)}. \end{aligned}$$

In the last line we have used both the generalised trace inequality, see [\[41, Sec. 2.5\]](#), and the Banach triple identification for the boundary terms. Similarly for \mathcal{D}_i^B :

$$\left| \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{D}_i^B(\mathbf{w}, \mathcal{B}_i, \eta_i, v_i) \right| \leq c \left(|B_i|_{C^1(\overline{\mathcal{Q}}_i; \mathbb{R}^d)}, |\nabla \cdot \mathbf{w}|_{C^1(\overline{\Omega}; \mathbb{R}^d)} \right) (\|\eta\|_{H(t)} + \|\eta\|_{\mathcal{V}_\Gamma^*(t)}) \|v\|_{Z_1(t)}.$$

Combining the previous three estimates we see

$$|b(t; \eta, v)| \leq c(\|\eta\|_{H(t)} + \|\eta\|_{\mathcal{V}_\Gamma^*(t)}) \|v\|_{Z_1(t)}. \quad (5.26)$$

In order to complete the proof, we employ the same duality argument as in [\[16, 22\]](#) to estimate the dual norm of η . Set $\mathcal{T} : \mathcal{H}_\Gamma(t) \rightarrow \mathcal{V}_\Gamma(t)$ as:

$$(\mathcal{T}\zeta, v)_{\mathcal{V}_\Gamma(t)} := \langle \zeta, v \rangle_{\mathcal{V}_\Gamma(t)} = (\zeta, v)_{\mathcal{H}_\Gamma(t)} \text{ for all } \zeta, v \in \mathcal{V}_\Gamma(t),$$

i.e \mathcal{T} acts a Riesz map mapping to the element in $\mathcal{V}_\Gamma(t)$ that corresponds to the functionals in $\mathcal{H}_\Gamma(t) \subset \mathcal{V}_\Gamma^*(t)$. Notice that:

$$\|\mathcal{T}\zeta\|_{\mathcal{V}_\Gamma(t)}^2 = \|\zeta\|_{\mathcal{V}_\Gamma^*(t)}^2 = \int_{\Gamma(t)} \zeta \mathcal{T}\zeta,$$

for any $\zeta \in \mathcal{H}_\Gamma(t)$. We note that substituting $v = \mathcal{T}(\zeta)$ in [\(5.19\)](#) gives:

$$\|\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\zeta)\|_{Z_1(t)} \leq c\|\mathcal{T}(\zeta)\|_{\mathcal{V}_\Gamma(t)} = c\|\zeta\|_{\mathcal{V}_\Gamma^*(t)} = c(\zeta, \mathcal{T}\zeta)_{\mathcal{H}_\Gamma(t)}^{1/2}. \quad (5.27)$$

By construction:

$$\begin{aligned}\|\eta\|_{\mathcal{V}_\Gamma^*(t)}^2 &= \int_{\Gamma(t)} \eta \cdot \mathcal{T}\eta = a_\kappa(t; \eta, \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)) \\ &= a_\kappa(t; \eta, \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta) - I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)]) + a_\kappa(t; \eta, I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)]).\end{aligned}$$

Then, for the first part, we have:

$$\begin{aligned}|a_\kappa(t; \eta, \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta) - I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)])| &\leq \|\eta\|_{V(t)} \|\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta) - I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)]\|_{V(t)} \\ &\leq ch \|\eta\|_{V(t)} \|\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)\|_{Z_1(t)} \leq ch \|\eta\|_{V(t)} \|\eta\|_{\mathcal{V}_\Gamma^*(t)}.\end{aligned}$$

For the second part, using the definition of the Ritz projection (5.8), we have

$$\begin{aligned}|a_\kappa(t; \eta, I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)])| &= |a_\kappa(t; \pi^h w, I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)]) - a_\kappa^h(t; \Pi^h w, I^h[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)])| \\ &\leq |a_\kappa(t; \pi^h w - w, I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)]) - a_\kappa^h(t; \Pi^h w - w^{-l}, I^h[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)])| \\ &\quad + |a_\kappa(t; w, I^l[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)] - \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)) - a_\kappa^h(t; w^{-l}, I^h[\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)^{-l}] - \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)^{-l})| \\ &\quad + |a_\kappa(t; w, \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)) - a_\kappa^h(t; w^{-l}, \mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)^{-l})| \\ &\leq ch^{k+1} \|w\|_{Z_1(t)} \|\mathcal{R}_{\mathcal{H}_\Gamma}(\mathcal{T}\eta)\|_{V(t)} + ch^{k+1} \|w\|_{V(t)} \|\eta\|_{\mathcal{V}_\Gamma^*(t)} + ch^{k+1} \|w\|_{Z_1(t)} \|\eta\|_{\mathcal{V}_\Gamma^*(t)} \\ &\leq ch^{k+1} \|w\|_{Z_1(t)} \|\eta\|_{\mathcal{V}_\Gamma^*(t)},\end{aligned}$$

using Lem. 3.10 and 5.8, the regularity estimate, (5.27), and (P1), (P4) and (P4'). Hence, we infer that

$$\|\eta\|_{\mathcal{V}_\Gamma^*(t)} \leq ch \|\eta\|_{V(t)} + ch^{k+1} \|w\|_{Z_1(t)}.$$

Substituting the final inequality here in (5.26) yields (5.25). \square

Lemma 5.8 ([22], Lem. 3.8 and 3.10). *For $w \in C_{Z_k}^1$, if (P1) to (P9), Lem. 3.10 and 5.6 and (5.25) hold, there exists $c > 0$ (independent of h) such that:*

$$\begin{aligned}\|\partial_t^h \Pi^h w\|_{V^h(t)} &\leq c(\|w\|_{V(t)} + \|\partial_t^\bullet w\|_{V(t)}), \\ \|w - \pi^h w\|_{H(t)} + h\|w - \pi^h w\|_{V(t)} &\leq ch^{k+1} \|w\|_{Z_k(t)}, \\ \|\partial_t^l(w - \pi^h w)\|_{H(t)} + h\|\partial_t^l(w - \pi^h w)\|_{V(t)} &\leq ch^{k+1} (\|w\|_{Z_k(t)} + \|\partial_t^\bullet w\|_{Z_k(t)}).\end{aligned}$$

Lemma 5.9. *The error estimates described in Lem. 5.8 also hold for a.e $t \in I$ for $w \in W(Z_k, Z_k)$.*

Proof. Via Lem. 2.7, $C_{Z_k}^1$ is dense within $W(Z_k, Z_k)$. For a $w \in W(Z_k, Z_k)$, take a sequence $w_\epsilon \in C_{Z_k}^1$ such that $w_\epsilon \rightarrow w$ in $W(Z_k, Z_k)$. This implies $\|w_\epsilon\|_{Z_k(t)} \rightarrow \|w\|_{Z_k(t)}$, $\|\partial_t^\bullet w_\epsilon\|_{Z_k(t)} \rightarrow \|\partial_t^\bullet w\|_{Z_k(t)}$, $\|\pi^h w_\epsilon\|_{V(t)} \rightarrow \|\pi^h w\|_{V(t)}$, $\|\partial_t^l \pi^h w_\epsilon\|_{V(t)} \rightarrow \|\partial_t^l \pi^h w\|_{V(t)}$ and $\|\partial_t^h \Pi^h w_\epsilon\|_{V^h(t)} \rightarrow \|\partial_t^h \Pi^h w\|_{V^h(t)}$ (by continuity of Π^h and π^l , see Rem. 5.5) in $L^2(I)$. We will only show the first inequality, but the same method can be used to obtain the two remaining ones. For arbitrary $\delta > 0$, there exists an $\epsilon_* > 0$ such that when $\epsilon > \epsilon_*$, we have:

$$\begin{aligned}\int_I (\|\partial_t^h \Pi^h(w)\|_{V^h(t)} - \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(t)})^2 dt &\leq \delta^2, \\ \int_I (\|\partial_t^\bullet w\|_{V(t)} - \|\partial_t^\bullet w_\epsilon\|_{V(t)})^2 dt &\leq \delta^2, \\ \int_I (\|w\|_{V(t)} - \|w_\epsilon\|_{V(t)})^2 dt &\leq \delta^2.\end{aligned}$$

We fix an arbitrary $t \in I$ and by taking the mean integral in $[t-s, t+s]$, $s > 0$ (reflecting the functions for $t < 0$, i.e. $v(t) = v(-t)$), we see:

$$\frac{1}{2s} \int_{t-s}^{t+s} \|\partial_t^h \Pi^h(w)\|_{V^h(\tau)} d\tau \leq \frac{1}{2s} \int_{t-s}^{t+s} \left| \|\partial_t^h \Pi^h(w)\|_{V^h(\tau)} - \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(\tau)} \right| + \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(\tau)} d\tau. \quad (5.28)$$

We note that via Young's inequality, (5.28) can be bounded by:

$$\begin{aligned} \frac{1}{2s} \int_{t-s}^{t+s} \left| \|\partial_t^h \Pi^h(w)\|_{V^h(\tau)} - \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(\tau)} \right| d\tau \\ \leq s^{-1/2} \left(\int_I \left(\|\partial_t^h \Pi^h(w)\|_{V^h(t)} - \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(t)} \right)^2 d\tau \right)^{1/2} \leq \delta s^{-1/2}. \end{aligned}$$

Whereas for the second term, we use the bounds given by Lem. 5.8 and Hölder's inequality once more:

$$\begin{aligned} \frac{1}{2s} \int_{t-s}^{t+s} \|\partial_t^h \Pi^h(w_\epsilon)\|_{V^h(\tau)} d\tau &\leq \frac{c}{2s} \int_{t-s}^{t+s} \|w_\epsilon\|_{V(\tau)} + \|\partial_t^\bullet w_\epsilon\|_{V(\tau)} d\tau \\ &\leq \frac{c}{2s} \int_{t-s}^{t+s} \|w\|_{V(\tau)} + \|\partial_t^\bullet w\|_{V(\tau)} + \left| \|w_\epsilon\|_{V(\tau)} - \|w\|_{V(\tau)} \right| \\ &\quad + \left| \|\partial_t^\bullet w_\epsilon\|_{V(\tau)} - \|\partial_t^\bullet w\|_{V(\tau)} \right| d\tau \\ &\leq \frac{c}{2s} \int_{t-s}^{t+s} \|w\|_{V(\tau)} + \|\partial_t^\bullet w\|_{V(\tau)} d\tau + 2\delta. \end{aligned}$$

Combining both of these estimates and equation (5.28):

$$\frac{1}{2s} \int_{t-s}^{t+s} \|\partial_t^h \Pi^h(w)\|_{V^h(\tau)} d\tau \leq \frac{c}{2s} \int_{t-s}^{t+s} \|w\|_{V(\tau)} + \|\partial_t^\bullet w\|_{V(\tau)} d\tau + (2 + s^{-1/2})\delta. \quad (5.29)$$

Since δ is arbitrary, letting $\delta = o(s^{1/2})$, we see that by the limit as $s \rightarrow 0$, via the Lebesgue differentiation theorem, both sides of equation (5.29) converge to their point-wise values a.e, hence we obtain that for all $t \in I$:

$$\|\partial_t^h \Pi^h(w)\|_{V^h(t)} \leq c(\|w\|_{V(t)} + \|\partial_t^\bullet w\|_{V(t)})$$

where the constant c is the same as the constant in its equivalent estimate in Lem. 5.8 (and hence independent of both h and t). \square

5.3. Proof of error bound.

Proof of Thm. 5.1. We have an additional right-hand side functional term that is not present in the original proof by Elliott and Ranner. This requires a modification of the proof of [22, Thm. 3.11].

To begin, we slightly modify the problem. For a test function $v \in W(V, H)$, we can rewrite the weak formulation of our problem as:

$$\frac{d}{dt} m(t; u, v) - m(t; u, \partial_t^\bullet v) + a(t; u, v) = l(t; v).$$

Employing the standard parabolic rescaling $\tilde{u} = e^{-\kappa t} u$ where κ is chosen as in the definition of the Ritz projection, the problem becomes:

$$\frac{d}{dt} m(t; \tilde{u}, v) - m(t; \tilde{u}, \partial_t^\bullet v) + a_\kappa(t; \tilde{u}, v) = \underbrace{e^{-\kappa t} l(v)}_{=: \tilde{l}(v)}. \quad (5.30)$$

Performing the same transformation to the discrete analogue: define $\check{U}_h = e^{-\kappa t} U_h$ which satisfies

$$\frac{d}{dt} m^h(t; \check{U}^h, v^h) - m^h(t; \check{U}, \partial_t^h v^h) + a_\kappa^h(t; \check{U}^h, v^h) = \underbrace{e^{-\kappa t} l^h(v^h)}_{=: \tilde{l}^h(v^h)}. \quad (5.31)$$

Set $\theta := \check{U} - \Pi^h \check{u}$, then using (5.30) and using the fact that $l^h(t; \cdot)$ equals $l(t; (\cdot)^l)$ for functions in $H^h(t)$, for arbitrary $v^h \in W(\mathcal{S}^h, \mathcal{S}^h)$, we arrive at:

$$\begin{aligned} & \frac{d}{dt} m^h(t; \Pi^h \check{u}, v^h) + a_\kappa^h(t; \Pi^h \check{u}, v^h) - m^h(t; \Pi^h \check{u}, \partial_t^h v^h) - \tilde{l}^h(v^h), \\ &= \frac{d}{dt} m^h(t; \Pi^h \check{u}, v^h) + a_\kappa(t; \check{u}, v^{h,l}) - m^h(t; \Pi^h \check{u}, \partial_t^h v^h) - \tilde{l}^h(v^h), \\ &= \frac{d}{dt} \left[m^h(t; \Pi^h \check{u}, v^h) - m(t; \check{u}, v^{h,l}) \right] - \left[m^h(t; \Pi^h \check{u}, \partial_t^h v^h) - m(t; \check{u}, \partial_t^\bullet v^{h,l}) \right], \\ &= m^h(t; \partial_t^h \Pi^h \check{u}, v^h) - m(t; \partial_t^l \check{u}, v^{h,l}) + \lambda^h(t; \Pi^h \check{u}, v^{h,l}) - \lambda^l(t; \check{u}, v^{h,l}) + m(t; \check{u}, \partial_t^\bullet v^{h,l} - \partial_t^l v^{h,l}). \end{aligned}$$

Now subtracting this equation from (5.31), and rearranging yields:

$$\begin{aligned} & \frac{d}{dt} m^h(t; \theta, v^h) + a_\kappa^h(t; \theta, v^h) - m^h(t; \theta, \partial_t^h v^h) \\ &= - \left[m^h(t; \partial_t^h \Pi^h \check{u}, v^h) - m(t; \partial_t^l \pi^h \check{u}, v^{h,l}) + m(t; \partial_t^l [\pi^h \check{u} - \check{u}], v^{h,l}) + \lambda^h(t; \Pi^h \check{u}, v^{h,l}) - \lambda^l(t; \pi^h \check{u}, v^{h,l}) \right. \\ & \quad \left. + \lambda^l(t; [\pi^h \check{u} - \check{u}], v^{h,l}) + m(t; \check{u}, \partial_t^\bullet v^{h,l} - \partial_t^l v^{h,l}) \right] =: -\Xi^h(\check{u}, v^h). \end{aligned} \quad (5.32)$$

Using the identity $\partial_t^l(v^{h,l}) = (\partial_t^h v^h)^l$ and looking at $\Xi^h(\cdot, \cdot)$ term by term, we see, for example:

$$\begin{aligned} & |m^h(t; \partial_t^h \Pi^h \check{u}, v^h) - m(t; \partial_t^l \pi^h \check{u}, v^{h,l})| \\ &= |m^h(t; \partial_t^h \Pi^h \check{u}, v^h) - m(t; (\partial_t^h \Pi^h \check{u})^l, v^{h,l}) + m(t; (\partial_t^h \Pi^h \check{u})^l - \partial_t^l \pi^h \check{u}, v^{h,l})|, \\ &\leq ch^{k+1}(\|\check{u}\|_{Z_k(t)} + \|\partial_t^\bullet \check{u}\|_{Z_k(t)}) \|v^h\|_{V^h(t)}, \end{aligned}$$

by (P1) and Lem. 5.9. Similar rearrangement and the use of Lem. 3.10 with (P2), (P3), (P8) yields:

$$|\Xi^h(\check{u}, v^h)| \leq ch^{k+1}(\|\check{u}\|_{Z_k(t)} + \|\partial_t^\bullet \check{u}\|_{Z_k(t)}) \|v^h\|_{V^h(t)}.$$

Using (5.32) and substituting $v^h = \theta$, we obtain:

$$\frac{d}{dt} m^h(t; \theta, \theta) + a_\kappa^h(t; \theta, \theta) - m^h(t; \theta, \partial_t^h \theta) = \Xi^h(\theta, \theta).$$

Using the transport formula and the bound on Ξ^h :

$$\frac{1}{2} \frac{d}{dt} m^h(t; \theta, \theta) + a_\kappa^h(t; \theta, \theta) \leq -\frac{1}{2} \lambda^h(t; \theta, \theta) + ch^{k+1}(\|\check{u}\|_{Z_k(t)} + \|\partial_t^\bullet \check{u}\|_{Z_k(t)}) \|\theta\|_{V^h(t)}.$$

Integrating over time and using Young's and Grönwall's inequality:

$$\sup_{t \in I} \|\theta\|_{H^h(t)}^2 + \int_0^T \|\theta\|_{V^h(t)}^2 \leq c \|\theta\|_{H^h(0)}^2 + c \int_0^T \|\theta\|_{H^h(t)}^2 + ch^{2k+2} \int_0^T (\|\check{u}\|_{Z_k(t)}^2 + \|\partial_t^\bullet \check{u}\|_{Z_k(t)}^2).$$

Finally, using the decomposition:

$$\check{u} - \check{U}^{h,l} = \check{u} - \pi^h \check{u} + \pi^h \check{u} - \check{U}^{h,l}.$$

Using the previous bound, the fact that the lift is a diffeomorphism and the bound on the Ritz map, we finally obtain:

$$\begin{aligned} \sup_{t \in I} \|\tilde{u} - \tilde{U}^{h,l}\|_{H(t)}^2 + h^2 \int_0^T \|\tilde{u} - \tilde{U}^{h,l}\|_{V(s)}^2 ds &= \sup_{t \in I} \|\tilde{u} - \pi^h \tilde{u} + \theta^l\|_{H(t)}^2 + h^2 \int_0^T \|\tilde{u} - \pi^h \tilde{u} + \theta^l\|_{V(s)}^2 ds, \\ &\leq c \|u_0 - u_0^{h,l}\|_{H(0)} + h^{2k+2} c(C_u). \end{aligned}$$

Undoing the scaling $u = e^{\kappa t} \tilde{u}$ gives us the desired error bound. \square

6. NUMERICAL RESULTS

All numerical results are computed using the `firedrake` package [7, 8, 15, 44]. Simulation code is available in [43]. Results are computed on a sequence of meshes generated using `GMSH` [25] rather than successive refinement of a single mesh.

The main challenges in implementing the numerical scheme are:

- (1) Computing the initial geometry: We start with a piecewise linear geometry given by `GMSH`. The initial isoparametric domain is computed through an explicit parametrisation applying directly the method from [Sec. 3](#) efficiently using custom written C code. The evolution of the mesh is carried out simply by moving the initial Lagrange nodes according to the smooth, given velocity field.
- (2) Labelling and tracking different parts of the domain: Along side the geometry and topology of the mesh we must track labels which say which elements are in domain $\Omega_1^h(t)$ or $\Omega_2^h(t)$ and which facets are on $\Gamma^h(t)$. Once this is fixed for the initial domains $\tilde{\Omega}_1^h$ and $\tilde{\Omega}_2^h$, this information is passed between different times. `GMSH` provides physical tags to each element and facets which can be used to identify the different domains.

Efficient and accurate quadrature rules are used to perform element-wise integrals. Note that system matrices must be reassembled at each time step due to the evolution of the domain.

6.1. Time discretisation of advection-diffusion problem. We start from the spatial discretisation from [Sec. 4](#). We will apply a backward difference formula (BDF) time discretisation of order q , see [35] for more including analysis of a similar surface only problem. We take a partition of the time interval $0 = t_0 < t_1 < \dots < t_M = T$. For simplicity we assume that each time interval is of the same length: $\tau := t_j - t_{j-1}$ for $j = 1, 2, \dots, M$.

We use temporal interpolations of each domain at each time step to construct a sequence of triangulations $\mathcal{J}^h(t_j)$ each equipped with finite element spaces $\mathcal{S}^h(t_j)$ for $j = 0, 1, \dots, M$. We define the discrete velocity $W^j \in \mathcal{S}^h(t_j)^d$ by

$$W^j = \frac{1}{\tau} \sum_{l=0}^q \delta_l X^{j-l}, \quad (6.1)$$

where X^j are the positions of the Lagrange nodes of the triangulation at time t_j and $\{\delta_l\}_{l=0}^q$ are the backward difference formula weights of order q , determined from the relation:

$$\delta(\zeta) = \sum_{l=0}^q \delta_l \zeta^l = \sum_{l=1}^q \frac{1}{l} (1 - \zeta)^l. \quad (6.2)$$

The fully discrete problem is the time discretisation (4.2): Given starting values $U^0 \in \mathcal{S}^h(t_0)$, \dots , $U^{q-1} \in \mathcal{S}^h(t_{q-1})$, and data $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and l_h , for $j = q, \dots, M$, we wish to find $U^j \in \mathcal{S}^h(t_j)$ as the solution

of

$$\frac{1}{\tau} \sum_{l=0}^q \delta_l m^h(t^{j-l}; U^{j-l}, \chi_i^{j-l}) + a^h(t^j; U^j, \chi_i^j) = l_h(t^j; \chi_i^j) \quad \text{for all basis function } \chi_i^j \in \mathcal{S}^h(t_j), \quad (6.3)$$

where again δ_l are the BDF weights (6.2). Note that the first term on the left hand side is computed by summing over q different meshes to approximate the time derivative. Let $U^h(t_j) := U^j$ and let $e^h(t) := u - U^{h,l}(t)$, then we assume that a similar estimate as in [35, Thm. 5.3] and [17, Thm. 2.4] holds towards the BDF scheme (6.3) in supplement to Thm. 5.1:

$$\|e^h(t_n)\|_{L^2(\Omega)}^2 + h^2 \tau \sum_{k=1}^n \|\nabla e^h(t_k)\|_{L^2(\Omega)}^2 \leq c(\tau^{2q} + h^{2k+2}). \quad (6.4)$$

Remark 6.1. We recall that on the space $V(t)$, both the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H(t)}$ are equivalent. Moreover, we expect the methods in [35] and in [17] are generalisable in our case in order to prove (6.4).

6.2. Numerical examples of advection-diffusion problem. For $d = 2, 3$, let $\Omega = [-1, 1]^d$, for $t \in [0, T]$, we define the evolution of the domain through the flow map Φ_t given by:

$$\Phi_t(x) = x + \frac{|x|^{1/3} \prod_{i=1}^d (1 - x_i^2)}{0.5 \prod_{i=1}^d (1 - 4x_i^2/|x|)} \begin{cases} ((\alpha(t) - 1)x_1, (\beta(t) - 1)x_2) & \text{if } d = 2, \\ ((\alpha(t) - 1)x_1, (\beta(t) - 1)x_2, 0) & \text{if } d = 3, \end{cases}$$

for $\alpha(t) = 1 + 0.25 \sin(t)$ and $\beta(t) = 1 + 0.25 \cos(t)$. This is a special motion which ensures that nodes initially on $\partial\Omega$ do not move and the surface $\Gamma(t)$ is described by the level set function $\phi(\cdot, t)$ given by

$$\phi(\cdot, t) = \begin{cases} \frac{x_1^2}{\alpha(t)^2} + \frac{x_2^2}{\beta(t)^2} - \frac{1}{2} & \text{if } d = 2 \\ \frac{x_1^2}{\alpha(t)^2} + \frac{x_2^2}{\beta(t)^2} + x_3^2 - \frac{1}{2} & \text{if } d = 3. \end{cases}$$

We define $\Omega_1(t)$ as the interior of $\Gamma(t)$ and $\Omega_2(t) = \Omega \setminus \Omega_1(t)$.

We set the coefficients in the equation to be $\mathcal{A}_1 = 10\text{Id}$, $\mathcal{A}_2 = \text{Id}$, $\mathcal{B}_1 = 5\nabla x_1$, $\mathcal{B}_2 = -5\nabla x_1$, $\mathcal{C}_1 = 1$, $\mathcal{C}_2 = 10$ and note that they jump across the interface. We set the right hand side data such that the exact solution u is given by

$$u(x, t) = \sin(t) |\Phi(x)| \prod_{i=1}^d \sin(2\pi x_i).$$

This exact solution is globally continuous, smooth in each domain but is not differentiable across the interface. In order to simplify the implementation the right hand side data (l_h) is computed by taking interpolations of smooth data. We compute using isoparametric elements of order 1, 2, 3 on a sequence of given meshes. For order k discretisation in space we use BDF order $k + 1$ in time. The initial solution $U^0 = 0$ matches the exact solution at $t = 0$. The other starting values are computed using lower order BDF methods. For elements of order k we expect convergence of order $k + 1$ for the error at the final time, $u(T) - U^M$, in the $L^2(\Omega)$ norm and order k is the $H^1(\Omega)$ semi-norm. The results are shown in Fig. 6.1 for the cases $d = 2, 3$ respectively. The precise numerical values are shown in Tab. 6.1. We see that the numerical results support the analytical convergence results.

AUTHOR CONTRIBUTIONS

Conceptualization, investigation, formal analysis, writing: All. Software: TR. Supervision: CME. Writing (first draft): PS.

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
1.00000	1.00000	3.51296×10^{-1}	3.30187	—	—
6.66667×10^{-1}	5.00000×10^{-1}	9.51874×10^{-1}	4.43031	-2.458416	-0.725039
4.37747×10^{-1}	2.50000×10^{-1}	1.99238×10^{-1}	2.33192	3.717892	1.525683
2.40008×10^{-1}	1.25000×10^{-1}	9.02047×10^{-2}	1.44891	1.318576	0.791859
1.34513×10^{-1}	6.25000×10^{-2}	2.78846×10^{-2}	7.80431×10^{-1}	2.027605	1.068576

(A) Order 1, $d = 2$

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
1.00000	1.00000	3.05166×10^{-1}	3.29594	—	—
6.66667×10^{-1}	5.00000×10^{-1}	2.14128×10^{-1}	2.03484	0.873768	1.189433
4.37747×10^{-1}	2.50000×10^{-1}	3.93009×10^{-2}	8.61571×10^{-1}	4.030255	2.043065
2.40008×10^{-1}	1.25000×10^{-1}	6.51346×10^{-3}	2.89276×10^{-1}	2.990807	1.816030
1.34513×10^{-1}	6.25000×10^{-2}	9.24067×10^{-4}	8.43187×10^{-2}	3.372712	2.129106

(B) Order 2, $d = 2$

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
1.00000	1.00000	1.87468×10^{-1}	2.57196	—	—
6.66667×10^{-1}	5.00000×10^{-1}	7.90234×10^{-2}	8.93941×10^{-1}	2.130551	2.606354
4.37747×10^{-1}	2.50000×10^{-1}	7.35252×10^{-3}	1.97059×10^{-1}	5.645317	3.594764
2.40008×10^{-1}	1.25000×10^{-1}	5.83518×10^{-4}	4.39236×10^{-2}	4.216077	2.497725
1.34513×10^{-1}	6.25000×10^{-2}	5.75918×10^{-5}	2.73132×10^{-2}	3.999388	0.820506

(C) Order 3, $d = 2$

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
1.25000	1.00000	6.90651×10^{-1}	8.44875	—	—
8.66599×10^{-1}	5.00000×10^{-1}	6.57464×10^{-1}	7.69389	0.134429	0.255490
6.31590×10^{-1}	2.50000×10^{-1}	5.72385×10^{-1}	6.68398	0.438071	0.444822
3.33531×10^{-1}	1.25000×10^{-1}	3.05710×10^{-1}	4.43667	0.982256	0.641827
1.75870×10^{-1}	6.25000×10^{-2}	9.68233×10^{-2}	2.34095	1.796513	0.998992

(D) Order 1, $d = 3$

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
8.66599×10^{-1}	5.00000×10^{-1}	3.78772×10^{-1}	5.39342	—	—
6.31590×10^{-1}	2.50000×10^{-1}	1.59518×10^{-1}	2.89154	2.733729	1.970655
3.33531×10^{-1}	1.25000×10^{-1}	2.14372×10^{-2}	9.12177×10^{-1}	3.143326	1.806895
1.75870×10^{-1}	6.25000×10^{-2}	2.84564×10^{-3}	2.57151×10^{-1}	3.155269	1.978425

(E) Order 2, $d = 3$

h	τ	L^2 error	H^1 error	eoc(L^2 error)	eoc(H^1 error)
1.25000	1.00000	3.19819×10^{-1}	4.78209	—	—
8.66599×10^{-1}	5.00000×10^{-1}	1.27429×10^{-1}	2.73209	2.511984	1.528193
6.31590×10^{-1}	2.50000×10^{-1}	3.16946×10^{-2}	9.05805×10^{-1}	4.398523	3.489946
3.33531×10^{-1}	1.25000×10^{-1}	2.44323×10^{-3}	1.41707×10^{-1}	4.013792	2.905320
1.75870×10^{-1}	6.25000×10^{-2}	1.70045×10^{-4}	2.10221×10^{-2}	4.164150	2.981596

(F) Order 3, $d = 3$ TABLE 6.1. Results for advection-diffusion problem for $d = 2, 3$.

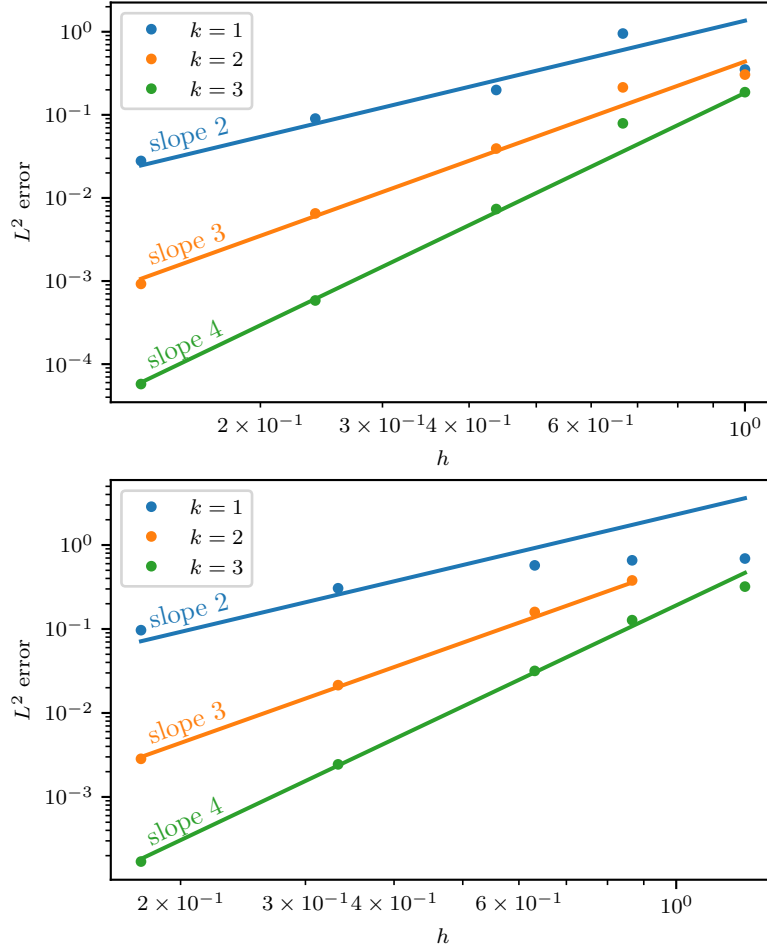


FIGURE 6.1. L^2 error for advection-diffusion problem for $d = 2$ (top) and $d = 3$ (bottom).

APPENDIX A. PROOF OF REGULARITY

In this section, we will show some results on the additional regularity of the smooth solution to (4.2).

Lemma A.1 (The Trace Map). *There exists a bounded and continuous linear operator $\tilde{\tau}_{(\cdot)} : L_V^2 \rightarrow L_{\mathcal{V}_\Gamma}^2$ such that $\tilde{\tau}_t \varphi(t) = \tau_t \varphi(t) \forall \varphi \in C_V$, where $\tau_t : V(t) \rightarrow \mathcal{V}_\Gamma(t)$ is the classical trace map.*

Proof. Let $\tau_t : V(t) \rightarrow \mathcal{V}_\Gamma(t)$ be the classical trace map. It is proven in [6] that the following identity: $\tau_t(\phi_t w_0) = \phi_t(\tau_0 w_0)$ holds for all $t \in I$ and $w_0 \in V(0)$. Moreover, there exists a c independent of time such that:

$$\|\tau_t(\phi_t w_0)\|_{\mathcal{H}_\Gamma(t)} \leq c \|\phi_t w_0\|_{\mathcal{V}_\Gamma(t)}.$$

Now, formally define $\tilde{\tau}_{(\cdot)}$ as:

$$\tilde{\tau}_{(\cdot)} v(\cdot) = \phi_{(\cdot)} \tau_0(\phi_{-(\cdot)} v(\cdot)).$$

Then via [Lem. 2.2](#), $\phi_{-(\cdot)}v(\cdot) \in L^2(I; V(0))$ and since τ_0 can also furthermore be uniquely identified as a linear map $\tau_0 : L^2(I; V(0)) \rightarrow L^2(I; \mathcal{V}_\Gamma(0))$, see [\[31, Thm. 1.2.4\]](#), finally the push-forward $\phi_t : L^2(I; \mathcal{V}_\Gamma(0)) \rightarrow L^2_{\mathcal{V}_\Gamma}$ maps back into the evolving space. Note that this map, by compatibility and it's time independent bound is also bounded. Finally, if $\varphi \in C_V$, then, at time $t \in I$:

$$\tilde{\tau}_t \varphi(t) = \phi_t \tau_0(\phi_{-t} \varphi(t)) = \tau_t \varphi(t). \quad \square$$

This allows us to formally identify the following pairing:

$$\int_0^T \langle g, v \rangle_{\mathcal{V}_\Gamma(t)} := \int_0^T \langle g, \tilde{\tau}_t v \rangle_{\mathcal{V}_\Gamma(t)}, \quad (g, v) \in L^2_{\mathcal{V}_\Gamma^*} \times L^2_V.$$

Lemma A.2. *Under the assumptions of [A4 Thm. 2.16](#), for each $g \in L^2_{\mathcal{V}_\Gamma}$, there exists a unique solution $u_g \in L^2_{Z_1}$ to:*

$$\sum_{i=1}^2 \int_0^T \int_{\Omega_i(t)} \mathcal{A}_i(t; x) \nabla u_g \cdot \nabla v = \int_0^T \int_{\Gamma(t)} g v \quad \text{for all } v \in L^2_V. \quad (\text{A.1})$$

Proof. It follows from the regularity assumptions on the flow Φ_t that the pair $(Z_1(t), \phi_t|_{Z_1(t)})|_{t \in I}$ is compatible. For $g \in L^2_{\mathcal{V}_\Gamma}$, we can take a subset $I' \subset I$ of full measure such that $g(t) \in \mathcal{V}_\Gamma(t)$ and $\|g(t)\|_{\mathcal{V}_\Gamma(t)} < \infty$, for $t \in I'$. Indeed via [Lem. 2.2](#), we can take the set of Lebesgue points of $\tilde{g}(\cdot) = \phi_{-(\cdot)}g(\cdot)$ to be I' and push the function forwards, $g(t) = \phi_t \tilde{g}(t) \in \mathcal{V}_\Gamma(t)$ for all $t \in I'$. Moreover, we see that, for $t \in I'$, these remain Lebesgue Points of $\|g(\cdot)\|_{\mathcal{V}_\Gamma(\cdot)}$ in $L^2(I)$, indeed, via the reverse triangle inequality and compatibility:

$$\begin{aligned} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \|g\|_{\mathcal{V}_\Gamma(s)} - \|g\|_{\mathcal{V}_\Gamma(t)}|^2 ds &\leq \frac{c}{2\delta} \int_{t-\delta}^{t+\delta} \|\tilde{g}(s)\|_{\mathcal{V}_\Gamma(0)} - \|\tilde{g}(t)\|_{\mathcal{V}_\Gamma(0)}|^2 ds \\ &\leq \frac{c}{2\delta} \int_{t-\delta}^{t+\delta} \|\tilde{g}(s) - \tilde{g}(t)\|_{\mathcal{V}_\Gamma(0)}^2 ds \end{aligned}$$

Fix $t \in I'$, set $u_g^t \in V(t)$ to be the solution to:

$$\sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{A}_i(t; x) \nabla u_g^t \cdot \nabla v = \int_{\Gamma(t)} g v \quad \forall v \in V(t). \quad (\text{A.2})$$

We will drop the distinction between I' and I and just say for almost all $t \in I$, then via the Hilbert triple structure outlined in [Sec. 2.3](#), $g(t) \in \mathcal{V}_\Gamma(t) \subset \mathcal{H}_\Gamma(t) \subset \mathcal{V}_\Gamma^*(t)$. By [\[37, Thm. 1\]](#) we have that for almost all $t \in I$ there exists a unique solution $u_g^t(\cdot) \in Z_1(t)$ to [\(A.2\)](#).

Set $u_g(t; \cdot) := u_g^t(\cdot)$, we show that this solution is in-fact in $L^2_{Z_1}$. We will proceed as follows:

- (1) First show that $u_g \in L^2_V$. By [\[4, Lem. 2.14\]](#), it suffices to show first that $t \rightarrow (u_g, w)_{V(t)}$ is measurable for all $w \in L^2_V$ and then that $\|u_g\|_{L^2_V} < \infty$.
- (2) We then reuse this method, showing that $t \rightarrow (u_g, w)_{Z_1(t)}$ is measurable for all $w \in L^2_{Z_1}$ and $\|u_g\|_{L^2_{Z_1}} < \infty$, and hence $u_g \in L^2_{Z_1}$.
- (3) Finally, we show that u_g does indeed solve [\(A.1\)](#).

To show the measurability, since the eigenvalues of $\mathcal{A}_i(t; x)$ are bounded from both below and above independent of time, we can induce the equivalent inner product $(u, v)_{\tilde{V}(t)} := (\mathcal{A}(t; x) \nabla u, \nabla v)_{H(t)}$. Showing

measurability then follows as:

$$(u_g, v)_{\tilde{V}(t)} = \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{A}_i(t; x) \nabla u_g \cdot \nabla v = \int_{\Gamma(t)} gv = \int_{\Gamma(t)} \langle g, v \rangle_{\mathcal{V}_{\Gamma}(t)},$$

and since $(v, g) \in L_V^2 \times L_{\mathcal{V}_{\Gamma}^*}^2$, by [4, Lem. 2.14], the map $t \mapsto \langle g, v \rangle_{\mathcal{V}_{\Gamma}(t)}$ is measurable and hence so is $(u_g, v)_{\tilde{V}(t)}$. For the uniform bound, testing the differential equation (A.2) with $v = u_g$ and integrating in time, we have:

$$\|u_g\|_{L_V^2}^2 \leq C(\gamma) \|g\|_{L_{\mathcal{V}_{\Gamma}^*}^2},$$

via Young's and Poincaré's inequalities, so $u_g \in L_V^2$. Before moving on, note that for any fixed $t \in I$, since $u_g(t; \cdot) \in Z_1(t)$, we can integrate by parts (A.2), obtaining:

$$\int_{\Gamma(t)} \left[\mathcal{A}(t; x) \nabla u \cdot \nu_{\Gamma} \right] v - \sum_{i=1}^2 \int_{\Omega_i(t)} \nabla \cdot (\mathcal{A}_i(t; x) \nabla u_g) v = \int_{\Gamma(t)} gv, \quad (\text{A.3})$$

for $v \in C_0^\infty(\Omega)$. For $v \in C_0^\infty(\Omega_1(t)) \times C_0^\infty(\Omega_2(t))$, we see that (A.3) yields:

$$\sum_{i=1}^2 \int_{\Omega_i(t)} \nabla \cdot (\mathcal{A}_i(t; x) \nabla u_g) v = 0.$$

Since the space $C_0^\infty(\Omega_1(t)) \times C_0^\infty(\Omega_2(t))$ is dense in $H(t)$, we get that:

$$\nabla \cdot (\mathcal{A}_i(t; x) \nabla u_g) = 0 \text{ a.e.} \quad (\text{A.4})$$

By the Poincaré's inequality, we can endow $Z_1(t)$ with a more convenient equivalent inner product:

$$(v, w)_{Z_1(t)} = \sum_{i=1}^2 \int_{\Omega_i(t)} \Delta v \cdot \Delta w + \nabla v \cdot \nabla w.$$

Since $\mathcal{A}(t; x)$ is assumed to be differentiable, we introduce the equivalent inner product on $Z_1(t)$:

$$(\eta, v)_{\tilde{Z}_1(t)} := \sum_{i=1}^2 \int_{\Omega_i(t)} \nabla \cdot (\mathcal{A}_i(t; x) \nabla \eta_i) \nabla \cdot (\mathcal{A}_i(t; x) \nabla v_i) + (\nabla \mathcal{A}_i(t; x)) \cdot \nabla \eta_i (\nabla \mathcal{A}_i(t; x)) \cdot \nabla v_i + \nabla \eta_i \cdot \nabla v_i.$$

We will first show the following statement, let ∇^2 be the Hessian, then:

$$\int_{\Omega_i(t)} |\mathcal{A}_i \nabla^2 \eta_i|^2 \geq \gamma \int_{\Omega_i(t)} |\Delta \eta_i|^2,$$

where $\gamma > 0$ is the coercivity constant in Thm. 2.16. For fixed $t \in I$, since $\mathcal{A}_i(t; x)$ is non-singular and symmetric, there exists orthogonal matrices $\mathcal{P}_i(t; x)$, $\mathcal{P}_i^T(t; x)$ and diagonal matrix $\mathcal{D}_i(t; x) = [\lambda_k(t; x) \delta_{k,l}]_{k,l=1}^n$, the eigenvalues of $\mathcal{A}_i(t; x)$ (note that eigenvalues are continuous for a continuous matrix) such that $\mathcal{A}_i(t; x) = \mathcal{P}_i(t; x) \mathcal{D}_i(t; x) \mathcal{P}_i^T(t; x)$. Doing a change of coordinates, $x = \mathcal{P}_i y$ and letting $\tilde{\eta}_i(x) = \eta_i(\mathcal{P}_i(x))$, we see:

$$\mathcal{A}_i(t; x) \nabla^2 (\eta_i(x)) = \mathcal{D}_i(t; x) \Delta \tilde{\eta}_i(y)|_{y=\mathcal{P}_i^T x}.$$

Hence, since \mathcal{P} is orthogonal:

$$\begin{aligned} \int_{\Omega_i(t)} |\mathcal{A}_i(t; x) \nabla^2(\eta_i(x))|^2 dx &= \int_{\mathcal{P}_i^T(\Omega_i(t))} |\mathcal{D}_i(t; \mathcal{P}_i(y)) \Delta \tilde{\eta}_i(y)|^2 dy, \\ &\geq \min_{x \in \Omega_i(t)} \min_{k \in [1, n]} \lambda_k^2(t; \mathcal{P}_i^T(x)) \int_{\mathcal{P}_i^T(\Omega_i(t))} |\Delta \tilde{\eta}_i(y)|^2 dy, \\ &\geq \gamma^2 \int_{\mathcal{P}^T(\Omega_i(t))} |\Delta \tilde{\eta}_i(y)|^2 dy = \gamma^2 \int_{\Omega_i(t)} |\Delta \eta_i(x)|^2 dx. \end{aligned}$$

To show the equivalence of inner products $(\cdot, \cdot)_{\tilde{Z}_1(t)}$ and $(\cdot, \cdot)_{Z_1(t)}$, note:

$$\begin{aligned} \|\eta\|_{\tilde{Z}_1(t)}^2 &= \sum_{i=1}^2 \int_{\Omega_i(t)} |\nabla \cdot (\mathcal{A}_i(t; x) \nabla \eta_i)|^2 + |(\nabla \mathcal{A}_i(t; x)) \cdot \nabla \eta_i|^2 + |\nabla \eta_i|^2 \\ &= \sum_{i=1}^2 \int_{\Omega_i(t)} |\mathcal{A}_i(t; x) \nabla^2 \eta_i|^2 + 2\mathcal{A}_i(t; x) \nabla^2 \eta_i (\nabla \mathcal{A}_i(t; x)) \cdot \nabla \eta_i + 2|(\nabla \mathcal{A}_i(t; x)) \cdot \nabla \eta_i|^2 + |\nabla \eta_i|^2 \\ &\geq \sum_{i=1}^2 \int_{\Omega_i(t)} \frac{1}{2} |\mathcal{A}_i(t; x) \nabla^2 \eta_i|^2 + |\nabla \eta_i|^2 \geq \min \left\{ \frac{1}{2} \gamma^2, 1 \right\} \|\eta\|_{Z_1(t)}^2 \end{aligned}$$

and:

$$\|\eta\|_{\tilde{Z}_1(t)}^2 \leq c \left[|\mathcal{A}_i|_{C^1(\Omega_i(t))} \right] (\|\nabla^2 \eta_i\|_{H(t)}^2 + \|\nabla \eta_i\|_{H(t)}^2) \leq c \|\eta\|_{Z_1(t)}^2.$$

Hence, $(\cdot, \cdot)_{\tilde{Z}_1(t)}$ and $(\cdot, \cdot)_{Z_1(t)}$ are equivalent. Substituting $\eta = u_g$ in the new inner product, using (A.4), we arrive at:

$$(u_g, v)_{\tilde{Z}_1(t)} = \sum_{i=1}^2 \int_{\Omega_i(t)} (\nabla \mathcal{A}_i(t; x)) \cdot \nabla u_i (\nabla \mathcal{A}_i(t; x)) \cdot \nabla v_i + \nabla u_i \cdot \nabla v_i.$$

Since we already know that $u_g \in L_V^2$, both $((\nabla \mathcal{A}_i(t; x)) \cdot \nabla u_i, (\nabla \mathcal{A}_i(t; x)) \cdot \nabla v_i)_{H(t)}$ and $(\nabla u_i, \nabla v_i)_{H(t)}$ are measurable and hence the map $t \rightarrow (u_g, v)_{\tilde{Z}_1(t)}$ is measurable for all $v \in Z_1(t)$. For the bound, [37, Thm. 1] gives us a constant C_t (that depends on time) such that:

$$\|u_g\|_{Z_1(t)} \leq C_t \|g\|_{V_\Gamma(t)}.$$

Using a similar method as Lem. 2.12 and changing the variables, it follows that there exists $C_T > C_t$, $C_T < \infty$. Hence:

$$\int_0^T \|u_g\|_{Z_1(t)}^2 dt \leq C_T^2 \int_0^T \|g\|_{V_\Gamma(t)}^2 dt < \infty.$$

Hence $u_g \in L_{Z_1}^2$.

Finally, we have a solution to $u_g \in L_{Z_1}^2$ solving (A.2) a.e in time. Since a function $v \in L_V^2$ satisfies $v(t) \in V(t)$ a.e in time, testing with such a $v(t)$ and integrating both sides of (A.2) yields the desired solution. Uniqueness follows via coercivity. \square

Lemma A.3. *Under the assumption A4 from Thm. 2.16, the solution of (A.1) posses a weak material derivative $\partial_t^\bullet u_g \in L_V^2$ and satisfies the estimate:*

$$\|u_g\|_{W(Z_1, V)}^2 \leq c \|g\|_{W(V_\Gamma, V_\Gamma^*)}^2$$

Proof. We use the same method as in the proof of [Lem. 5.4](#). Let:

$$k(t; v, w) := \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{A}_i(t; x) \nabla v \cdot \nabla w$$

then we see, going back to [\(A.2\)](#), we see that u_g solves:

$$k(t; u_g, w) = (g, w)_{\mathcal{H}_\Gamma(t)},$$

a.e for all $w \in L_V^2$ (as before, we can take a subset of I of full measure such that $w(t) \in V(t)$, moreover, via the classical trace theorem, we can identify $w(t) \in \mathcal{H}_\Gamma(t)$ by $(g, w)_{\mathcal{H}_\Gamma(t)} = (g, \tau_t w)_{\mathcal{H}_\Gamma(t)}$). The derivative of $k(t; \cdot, \cdot)$, $\dot{k}(t; \cdot, \cdot)$, can be explicitly calculated to be:

$$\frac{d}{dt} k(t; w, v) = \dot{k}(t; w, v) + k(t; \partial_t^\bullet w, v) + k(t; w, \partial_t^\bullet v), \quad \forall v, w \in W(V, V), \quad (\text{A.5})$$

where

$$\dot{k}(t; w, v) = \sum_{i=1}^2 \int_{\Omega_i(t)} \mathcal{D}_i^A(\mathbf{w}, \mathcal{A}_i, w_i, v_i),$$

and \mathcal{D}_i^A was defined in [\(2.12\)](#). We set \tilde{u}_g to be the solution to:

$$\int_0^T k(t; \tilde{u}_g, \eta) dt = \int_0^T \lambda_\Gamma(t; g, \eta) - \dot{k}(t; u_g, \eta) + \langle \partial_t^\bullet g, \eta \rangle_{\mathcal{V}_\Gamma(t)} dt \quad \forall \eta \in L_V^2. \quad (\text{A.6})$$

The material derivative taken on the function $\partial_t^\bullet g$ is the one with the triple $L_{\mathcal{V}_\Gamma}^2 \subset L_{\mathcal{H}_\Gamma}^2 \subset L_{V_\Gamma^*}^2$ and the bilinear form λ_Γ is the corresponding form from [Def. 2.3](#) satisfying the equation:

$$\frac{d}{dt} (v, w)_{\mathcal{H}_\Gamma(t)} = \langle \partial_t^\bullet v, w \rangle_{\mathcal{V}_\Gamma(t)} + \langle \partial_t^\bullet w, v \rangle_{\mathcal{V}_\Gamma(t)} + \lambda_\Gamma(t; v, w).$$

See [\[5, Sec. 5.4\]](#) for an explicit form of $\lambda_\Gamma(t; \cdot, \cdot)$. Using the same method as the proof of [Lem. A.2](#), we have that if $\partial_t^\bullet g \in L_{\mathcal{V}_\Gamma^*}^2$, there exists a unique $\tilde{u}_g \in L_V^2$ solving equation [\(A.6\)](#).

Let u_g be the solution of [\(A.2\)](#). Via isomorphism, $\phi_{-(\cdot)} \tilde{u}_g(\cdot), \phi_{-(\cdot)} u_g(\cdot) \in L^2(I; V(0))$, we pick a Lebesgue point $s_* \in I$ of $\phi_{-(\cdot)} u_g(\cdot)$ and set

$$w := \phi_t \int_{s_*}^t \phi_{-\tau} \tilde{u}_g(\tau) d\tau + \phi_t \phi_{-s_*} u_g(s_*) = \phi_t \int_{s_*}^t \phi_{-\tau} \tilde{u}_g(\tau) d\tau + z_{s_*}.$$

Thus $w \in W(V, V)$ and $\partial_t^\bullet w = \tilde{u}_g$. We aim to show that $w = u_g$. To do so, note that by definition of $\dot{k}(t; \cdot, \cdot)$ [\(A.5\)](#), testing with $\eta \in W(V, V)$ with $\eta(0) = \eta(T) = 0$:

$$\int_0^T k(t; \partial_t^\bullet w, \eta) dt = \int_0^T -\dot{k}(t; w, \eta) - k(t; w, \partial_t^\bullet \eta) dt. \quad (\text{A.7})$$

we see from comparing [\(A.6\)](#) and [\(A.7\)](#):

$$-\int_0^T k(t; w, \partial_t^\bullet \eta) dt = \int_0^T \dot{k}(t; w - u_g, \eta) - (\partial_t^\bullet \eta, g)_{\mathcal{H}_\Gamma(t)} dt, \quad \forall \eta \in W(V, V), \quad \eta(0) = \eta(T) = 0. \quad (\text{A.8})$$

Comparing [\(A.8\)](#) with [\(A.2\)](#), we infer:

$$\int_0^T k(t; u_g - w, \partial_t^\bullet \eta) dt = \int_0^T \dot{k}(t; w - u_g, \eta) dt.$$

Letting $\eta(t) = \psi(t)v(t)$ where $\psi(t) \in \mathcal{D}(I)$ and $v(t) \in W(V, V)$, we see:

$$\int_0^T \psi(t) k(t; u_g - w, \partial_t^\bullet v) + \psi'(t) k(t; u_g - w, v) dt = \int_0^T \psi(t) \dot{k}(t; w - u_g, v) dt.$$

Since this holds for arbitrary $\psi \in \mathcal{D}(I)$, then, by use of [33, Lem. 1.2.1], there exists some $c \in \mathbb{R}$ such that:

$$k(t; u_g - w, v) = \int_{s_*}^t \dot{k}(\tau; w - u_g, v) + k(\tau; w - u_g, \partial_t^\bullet v) d\tau + c, \quad (\text{A.9})$$

a.e in time. Note that the right hand side of (A.9) is absolutely continuous in time and hence is the unique continuous representative of $k(t; u_g - w, v)$ (since $k(\cdot; u_g - w, v)$ is in $L^1(I)$ as a function of time). It also follows from the fact that $k(t; \eta, v)$ is continuous for $\eta, v \in C_V^0$ that s_* is also a Lebesgue point of $k(t; u_g - w, v)$ (by use of a standard density argument). Since the continuous representative equals its L^p counterpart on Lebesgue point (as Lebesgue points are also points of approximate continuity, see [23, Sec. 1.7]), evaluating both sides of (A.9) at $t = s_*$, using the definition of w , yields $c = 0$. Finally we can test with $\eta(s) = \phi_s \phi_{-t}(u_g - w)$ and by use of the same argument as Lem. 5.4 we see that $u_g = w$ and hence $u_g \in W(Z_1, V)$. \square

From the previous two lemmas, we can show a time regularity result for (4.2).

Lemma A.4. *Under the assumptions A1 to A4 in Thm. 2.16, the solution u to problem (2.8) posses the additional regularity $u \in W(V, H)$.*

Proof. Let $z = u - u_g$, where u is the weak solution from the problem in (2.8), then:

$$\int_0^T \langle \partial_t^\bullet z, v \rangle_{V(t)} + a(t; z, v) + \lambda(t; z, v) = \int_0^T \tilde{l}(t; v),$$

where:

$$\tilde{l}(t; v) = (f, v)_{H(t)} - (\partial_t^\bullet u_g, v)_{H(t)} - \lambda(t; u_g, v) - ([\mathcal{B} - \mathbf{w}] \cdot \nabla u_g - [\mathcal{C} - \nabla \cdot \mathbf{w}] u_g, v)_{H(t)}.$$

By the regularity of u_g , this is a functional in L_H^2 , by [4, Thm. 3.13], $z \in W(V, H)$ and hence so is $u \in W(V, H)$. \square

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